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PROBLEM OF THE ISOCHRONOUS HAIRSPRING

By

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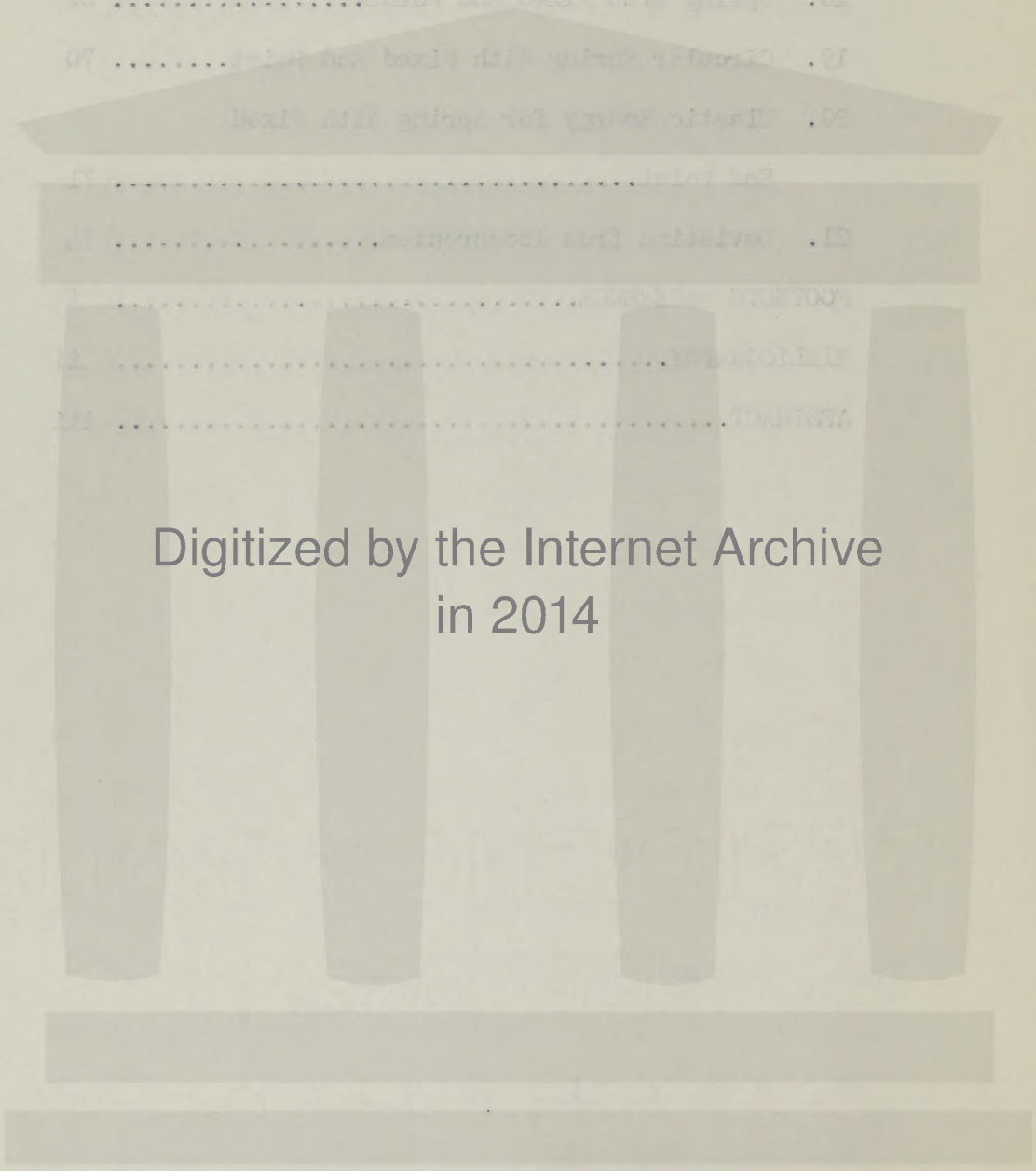

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I. THE PROBLEM

Introduction

The system used to regulate watches and chronometers consists of a hairspring having one end fastened to the axis of the balance wheel, and the other end fastened by means of a "stud" to the frame of the watch. The system does not perform an exact simple harmonic motion, and the period of vibrations depends on the amplitude of the balance wheel.

If the balance wheel is turned out of its position of equilibrium, a reaction force isset up in the two end points which causes a deviation from isochronism of the vibrations.

For the proper function of the watch, it is necessary to reduce the lack of isochronism of the vibrations as much as possible; and, if the shape of the spring is designed so that the reaction force at the end points is as small as possible, the deviation from isochronism will be correspondingly small.

The method used to solve this problem, in practical applications, is one developed by Phillips.*(5) He found that for a spring fastened to the balance axis and "stud" the reaction force at the ends cannot be made zero, but can

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The method used to solve this problem, in practical applications, is one developed by Phillips.*(2) He found that for a spring fastened to the balance axis and "stud" the reaction force at the ends cannot be made zero, but can

be made very small by designing the end portion or "terminals" of the spring to have special shapes.

Resale*(4) and Caspari*(1) found that it is possible to design the shape of the hairspring so that the reaction force changes the period by an amount which is approximately independent of the amplitude. For this solution the reaction force need not be small.

Bouasse*(1) proposed by means of a model that perfect isochronism would result for a flat spiral spring if the outer end point were not fixed, but free to move in a plane with a condition that the tangent at the end point remains the same. This method remains in theory only, because it cannot be realized in a watch. Bouasse illustrated this by fixing the outer end point of the spring to a rod which can move as a spherical pendulum (fig. 2). If the pendulum is balanced by a counter weight at the top, the end point will move parallel to itself and will not create a reaction force on the end of the spring. This arrangement constitutes a free spring.

The outer end point will experience a parallel displacement if the inner end point is turned over a certain angle, and this displacement is proportional to the reaction force that would exist at the end point if it were fixed

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The outer end point will experience a parallel displacement if the inner end point is turned over a certain angle, and this displacement is proportional to the reaction force that would exist at the end point if it were fixed

with the stud.

The most rigorous mathematical treatment of the problem has been given by S. Goudsmit and M. Wang. This method of development is based on the method described very briefly by J. Haag of the Institut de Chronometrie at Besancon, France.

The present paper is a survey of the methods of solution proposed by Phillips, Resal, and Goudsmit and Wang. The original works of Phillips could not be located; however, an excellent description is given by H. Bouasse*(1) in Pendule Spiral Diapason. The methods of Phillips and Resal were translated and rephrased from the French, and much of the mathematics carried out in detail.

The greater part of the work is based entirely on the method of Goudsmit and Wang with all of the mathematics carried out in detail. In some instances the phrasing of the authors has been kept for clarity of expression. Finally, the important resulting formulas were applied to original specific shapes, characterizing the shape of a hairspring.

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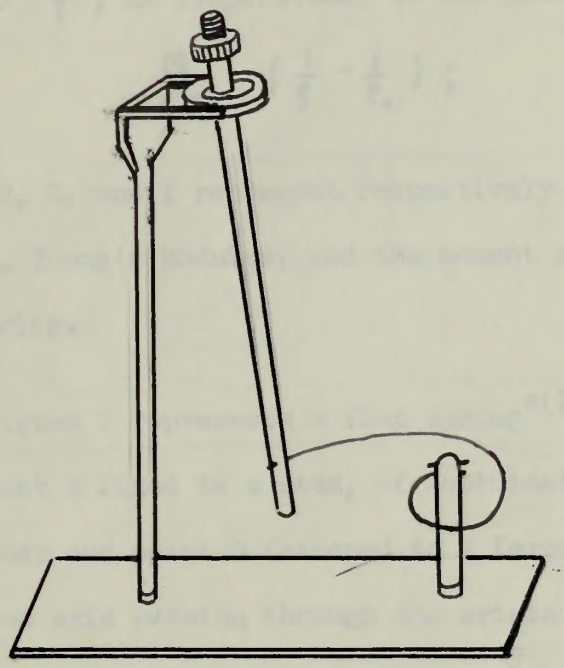
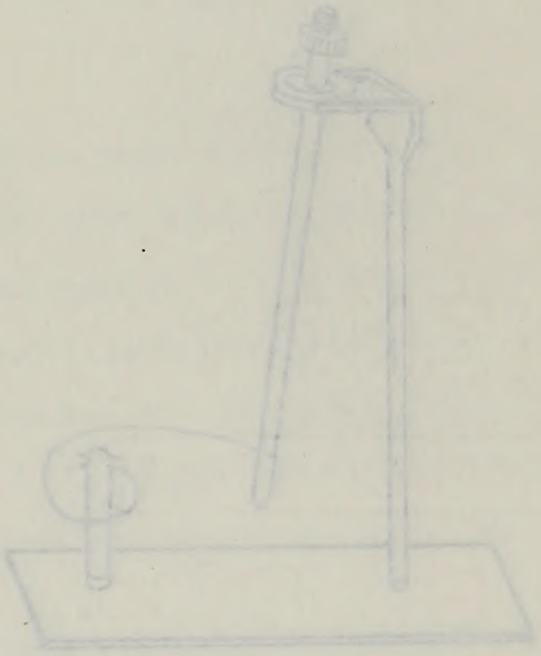


BOUASSE PENDULUM

FIGURE 1.



II. MATHEMATICAL

1. Analytic Representation of a Flat Spring

In accordance with the theory of elasticity, the couple required to deform an element of a spring, in a manner such that the radius of curvature changes from ρ_o to ρ, is proportional to the quantity:

$$\frac{M}{EI} = \left(\frac{1}{\rho} - \frac{1}{\rho_o}\right); \qquad (1)$$

where M, E, and I represent respectively the bending moment, Young's Modulus, and the moment of inertia of the spring.

Figure 2 represents a flat spring*(1) having one end point A fixed to a stud, of coordinates X_oY_o, and the other end point B fastened to a ferrule which rotates about an axis passing through the origin of the coordinate system, and perpendicular to the xy plane. The fixed position of the stud is equivalent to the application, at the extremity A, of a couple C and a force F with components F_x, F_y.

To analyze the forces acting on the spring, it is assumed that the spring terminates at an arbitrary point P, of coordinates x, y. At this point is applied a couple D and a force having components $-F_x$, $-F_y$. For equilibrium, the sum of all the external forces and the couples must be

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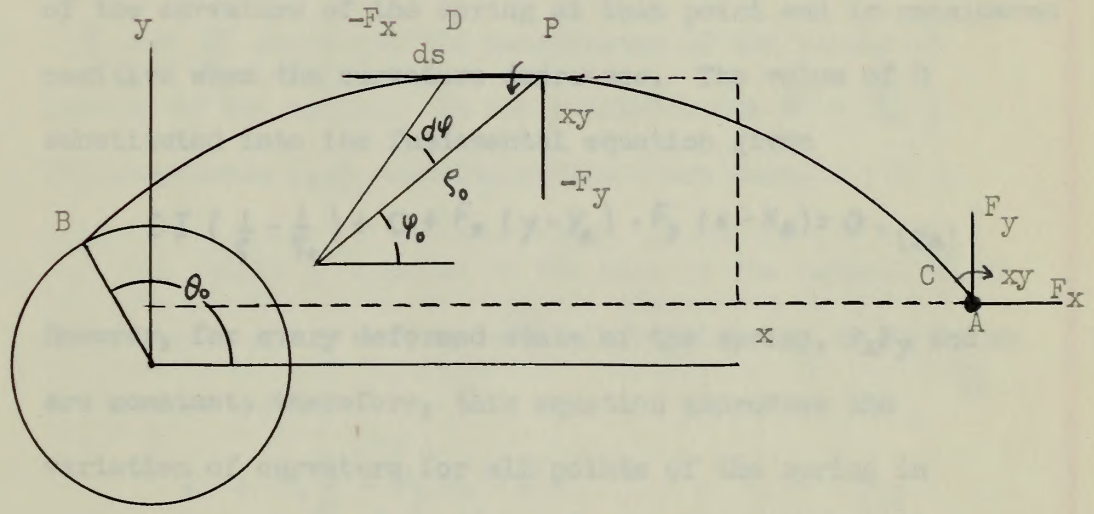
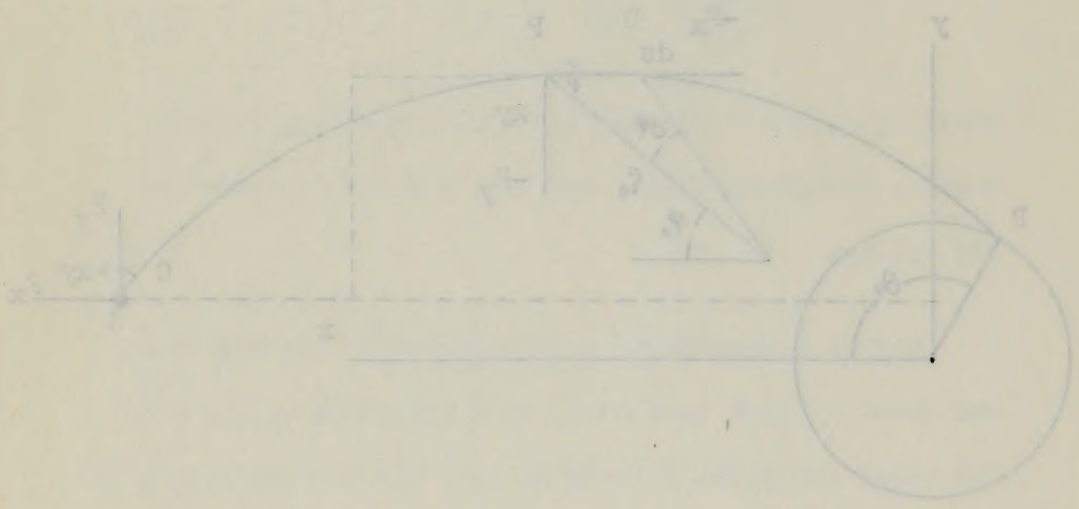


FIGURE 2. FLAT SPRING



equal to zero. The couple produced by the components F_x, $-F_x$ is $F_x\,(y-y_o)$ and that produced by the components F_y, $-F_y$ is $F_y\,(x_o-x)$. It follows that

$$D + C + F_x(y-y_o) - F_y(x-x_o) = 0. \qquad (2)$$

The couple D at the Point P is measured by the variation of the curvature of the spring at that point and is considered positive when the curvature decreases. The value of D substituted into the fundamental equation gives

$$EI\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right) + C + F_x(y-y_o) - F_y(x-x_o) = 0. \qquad (2a)$$

However, for every deformed state of the spring, F_xF_y and C are constant; therefore, this equation expresses the variation of curvature for all points of the spring in changing from the non-deformed to the deformed state.

φ_o and φ are the angles that the radius of curvature, to the point P, makes with the x axis for the spring in the non-deformed and the deformed state.

Then
$$ds = \rho_o\, d\varphi_o,$$
$$\text{and} \quad ds = \rho\, d\varphi. \qquad (3)$$

If equation (2a) is multiplied by ds and integrated over the length of the Spring L, the resulting equation is

ρ_0 and φ are the angles that the radius of curvature, to the point P, makes with the x axis for the spring in the non-deformed and the deformed state.

Then $ds = \rho_0 \, d\varphi_0$.

and $ds = \rho \, d\varphi$. (3)

If equation (2a) is multiplied by ds and integrated over the length of the spring L , the resulting equation is

$$\frac{EI}{L}(\varphi - \varphi_0) + F_x(\eta - y_0) - F_y(\xi - x_0) + C = 0, \tag{4}$$

in which $\frac{ds}{\rho_0}$ and $\frac{ds}{\rho}$ are replaced by φ_0 and φ respectively, and

$$L\xi = \int x\,ds\,; \qquad L\eta = \int y\,ds. \tag{5}$$

ξ and η represent the coordinates of the center of gravity of the spring. In the initial state $\varphi = \varphi_0$, the components F_x, F_y and the couple C are zero.

The couple Γ applied at the axis of the balance wheel (coordinates x = y = 0) to maintain equilibrium is contained in the equation

$$C = F_x y_0 - F_y x_0 - \Gamma. \tag{6}$$

The values C and $\varphi - \varphi_0 = \theta$ substituted into equation (4) gives

$$E\frac{I\theta}{L} + F_x\eta - F_y\xi = \Gamma. \tag{7}$$

The couple which the spring exerts on the balance wheel is equal and opposite to Γ.

$$\frac{E}{EI}(\psi-\psi_0)+F_x(\eta-\eta_0)-F_y(\xi-x_0)+C=0, \qquad (4)$$

in which $\frac{\psi_0}{ds}$ and $\frac{\psi}{ds}$ are replaced by ψ_0 and ψ respectively, and

$$E\xi=\int x\,ds\,; \qquad E\eta=\int y\,ds. \qquad (5)$$

ξ and η represent the coordinates of the center of gravity of the spring. In the initial state $\psi=\psi_0$, the components F_x, F_y and the couple C are zero.

The couple L applied at the axis of the balance wheel (coordinates $x=y=0$) to maintain equilibrium is contained in the equation

$$C=F_x\eta_0-F_y x_0-L. \qquad (6)$$

The values C and $\psi-\psi_0=\Theta$ substituted into equation (4) gives

$$\frac{E}{EI}\Theta+F_x\eta-F_y\xi=L. \qquad (7)$$

The couple which the spring exerts on the balance wheel is equal and opposite to L.

The equation of motion of the balance wheel is now

$$I\frac{d^2\theta}{dt^2} + \frac{EI\theta}{L} + F_x\eta - F_y\xi = 0; \qquad (8)$$

in which $I\frac{d^2\theta}{dt^2} = -\Gamma$, and I is moment of inertia of the balance wheel. The oscillations are isochronous only if $F_x\eta$ and $F_y\xi$ are zero. Hence,

$$I\frac{d^2\theta}{dt^2} + \frac{EI\theta}{L} = 0. \qquad (9)$$

This expresses that the oscillations are isochronous only if the reaction force or the pressure at the stud is zero. In the case of a spiral, the center of gravity of the spiral must coincide with the axis of the balance wheel for isochronism.

The equation of motion of the balance wheel is now

$$I \frac{d^2\theta}{dt^2} + \frac{E}{L} I\theta + F_x \eta - F_y \xi = 0 ; \qquad (8)$$

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Hence,

$$I \frac{d^2\theta}{dt^2} + \frac{E}{L} I\theta = 0 . \qquad (9)$$

This expresses that the oscillations are isochronous only if the reaction force or the pressure at the stud is zero. In the case of a spiral, the center of gravity of the spiral must coincide with the axis of the balance wheel for isochronism.

2. First Problem by Phillips

The deviation from isochronism can be made small if the shape of the spring is designed such that the reaction force at the end points is small. The reaction force at the end point is proportional to the displacement of the end point that would occur for a free spring; however, Phillips' solution to the flat spring primarily expresses this. It is based on the problem of finding a possible curve such that the terminal point has a given trajectory or displacement, and simultaneously, at any point of the trajectory the tangent to the point has an imposed direction. It is required that the spring deforms in a manner as to satisfy the condition

$$\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) = \frac{M}{EI} = \beta .$$

The flat spring which is deformed*(1) has the outer end point A fastened to a stud, and the inner end point B fastened to a balance wheel of radius r and center O (fig. 3). In the initial state the tangent to the arbitrary point C, a function of the arc length S, makes an angle θ_0 with the y axis. The radius of curvature normal to the point makes the same angle θ_0 with the x axis.

Hence,

$$S = \rho_0 \theta_0 , \qquad (10)$$

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$$\left(\frac{1}{\rho} - \frac{1}{\rho_o} \right) = \frac{M}{EI} = \beta .$$

The flat spring which is deformed*(1) has the outer end point A fastened to a strut, and the inner end point B fastened to a balance wheel of radius r and center O (Fig. 3). In the initial state the tangent to the arbitrary point C, a function of the arc length s, makes an angle θ_o with the y axis. The radius of curvature normal to the point makes the same angle θ_o with the x axis.

Hence,

$$s = \rho_o \, \theta_o ,$$

(10)

and $S = \rho\,\theta$.

In the deformed state

$$\left(\frac{\theta}{S} - \frac{\theta_0}{S}\right) = -\beta,$$

or

$$\theta = \theta_0 - \beta S. \tag{11}$$

The constant $\frac{M}{EI} = \beta$ is chosen negative because the angle θ_0 decreases as the spring increases. The angle which the tangent makes with the end of any element of arc ds is φ_0 and φ for the non-deformed and the deformed states of the spring.

Hence,

$$ds = \varphi_0 \rho_0 = \varphi \rho,$$

and

$$\varphi - \varphi_0 = \left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) ds = \beta\, ds. \tag{12}$$

In the deformed state the tangent at the point A is invariable, but the angle through which the tangent at C turns is:

$$\int_0^S \beta\, ds = \beta S. \tag{13}$$

Therefore, $\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) = \beta$ expresses that following the deformation, the tangent to a point C of the spring

and $S = \rho\,\theta$.

In the deformed state

$$\left(\frac{\theta}{S} - \frac{\theta_o}{S}\right) = -\lambda .$$

or

$$\theta = \theta_o - \lambda S . \tag{11}$$

The constant $\lambda = \frac{M}{EI}$ is chosen negative because the angle θ_o decreases as the spring increases.

The angle which the tangent makes with the end of any element of arc ds is ψ_o and ψ for the non-deformed and the deformed states of the spring.

Hence,

$$d\,s = \psi_o\,\rho_o = \psi\,S ,$$

and

$$\psi - \psi_o = \left(\frac{1}{\rho} - \frac{1}{\rho_o}\right) ds = \lambda\, ds . \tag{12}$$

In the deformed state the tangent at the point A is invariable, but the angle through which the tangent at C turns is:

$$\int_o^s \lambda\, ds = \lambda\, S . \tag{13}$$

Therefore, $\left(\frac{1}{\rho} - \frac{1}{\rho_o}\right) = \lambda$ expresses that following the deformation, the tangent to a point C of the spring

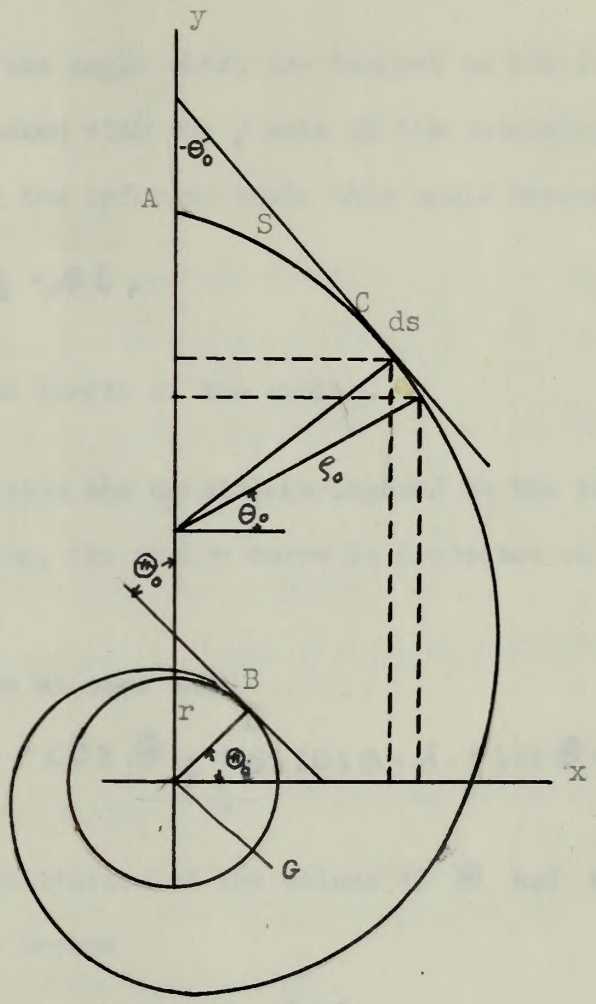
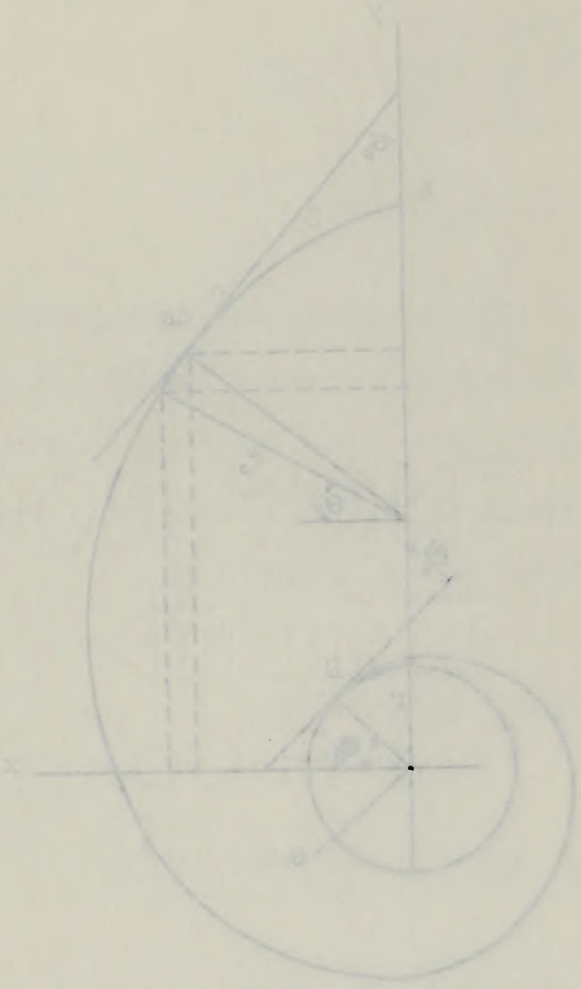


FIGURE 3. FLAT SPRING BY PHILLIPS



turns proportionally to its distance from the fixed point A.

Θ_0 is the angle which the tangent to the inner end point B makes with the y axis in the non-deformed state, and in the deformed state this angle becomes

$$\Theta = \Theta_0 - \beta L, \tag{14}$$

where L is the length of the spring.

To determine the conditions imposed on the extremity B of the spring, the entire curve is projected on to the x and y axis.

It can be seen at once that

$$\int_0^L ds \sin\theta = r\cos\Theta \; ; \int_0^L ds\cos\theta = \delta - r\sin\Theta. \tag{15}$$

Following substitution of the values of Θ and θ, the two equations become

$$\int_0^L ds \sin(\theta_0 - \beta s) = r\cos(\Theta_0 - \beta L) \; ; \tag{16}$$
$$\int_0^L ds\cos(\theta_0 - \beta s) = \delta - r\sin(\Theta_0 - \beta L).$$

The solution to the problem can exist only if these relations are satisfied indentically. This is generally impossible unless β is very small. If the balance wheel rotates through an angle α, then

$$\alpha = \Theta - \Theta_0 = \frac{ML}{EI}, \tag{17}$$

turns proportionally to its distance from the fixed point A.

Θ_0 is the angle which the tangent to the inner end point B makes with the y axis in the non-deformed state, and in the deformed state this angle becomes

$$\Theta = \Theta_0 - \beta L , \tag{14}$$

where L is the length of the spring.

To determine the condition imposed on the extremity B of the spring, the entire curve is projected on to the x and y axis.

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$$\alpha = \Theta - \Theta_0 = \frac{ML}{EI} , \tag{17}$$

and the movement of the balance wheel will be isochronous. With I_1 the moment of inertia of the balance wheel, the period*(4) is

$$T = \pi \sqrt{\frac{I_1 L}{E I}} . \tag{18}$$

By Taylor's expansion

$$\begin{aligned}
\cos(\theta_0 - \beta s) &= \cos\theta_0 + \beta s \sin\theta_0 + \frac{\beta^2 s^2}{2!}\cos^2\theta_0 + \cdots , \\
\sin(\theta_0 - \beta s) &= \sin\theta_0 - \beta s \cos\theta_0 + \frac{\beta^2 s^2}{2!}\sin^2\theta_0 + \cdots , \\
\cos(\Theta_0 - \beta L) &= \cos\Theta_0 + \beta L \sin\Theta_0 + \frac{\beta^2 L^2}{2!}\cos^2\Theta_0 + \cdots , \\
\sin(\Theta_0 - \beta L) &= \sin\Theta_0 - \beta L \cos\Theta_0 + \frac{\beta^2 L^2}{2!}\sin^2\Theta_0 + \cdots .
\end{aligned} \tag{19}$$

These equations substituted into (16) give

$$\int_0^L \{\cos\theta_0 + \beta s \sin\theta_0 + \cdots\} ds = \delta - r\sin\Theta_0 + r\beta L\cos\Theta_0 + \cdots ,$$

and (20)

$$\int_0^L \{\sin\theta_0 - \beta s\cos\theta_0 + \cdots\} ds = r\cos\Theta_0 + r\beta L\sin\Theta_0 + \cdots .$$

If the coefficients of the various powers of β are equated, the equations reduce to

$$\begin{aligned}
\int_0^L \cos\theta_0\, ds &= \delta - r\sin\Theta_0 , \\
\int_0^L \sin\theta_0\, ds &= r\cos\Theta_0 ,
\end{aligned} \tag{21}$$

$$-\int_0^L \cos\theta_0\, s\, ds = rL\sin\Theta_0 ,$$

and

$$\int_0^L \sin\theta_0\, s\, ds = rL\cos\Theta_0 . \tag{22}$$

and the movement of the balance wheel will be isochronous.

With I, the moment of inertia of the balance wheel, the period[*(1)] is

$$T = \pi \sqrt{\frac{I'L}{EI}} \,. \qquad (18)$$

By Taylor's expansion

$$\sin(\Theta^\circ - \sqrt{3}s) = \sin\Theta^\circ - \sqrt{3}s\cos\Theta^\circ + \frac{\sqrt{3}^2 s^2}{2!}\sin\Theta^\circ + \cdots ,$$

$$\cos(\Theta^\circ - \sqrt{3}s) = \cos\Theta^\circ + \sqrt{3}s\sin\Theta^\circ + \frac{\sqrt{3}^2 s^2}{2!}\cos\Theta^\circ + \cdots ,$$

$$\sin(\Theta^\circ - \sqrt{3}L) = \sin\Theta^\circ - \sqrt{3}L\cos\Theta^\circ + \frac{\sqrt{3}^2 L^2}{2!}\sin\Theta^\circ + \cdots ,$$

$$\cos(\Theta^\circ - \sqrt{3}L) = \cos\Theta^\circ + \sqrt{3}L\sin\Theta^\circ + \frac{\sqrt{3}^2 L^2}{2!}\cos\Theta^\circ + \cdots , \qquad (19)$$

These equations substituted into (16) give

$$\int_0^L \{\cos\Theta^\circ + \sqrt{3}s\sin\Theta^\circ + \cdots\}\, ds = \delta - r\sin\Theta^\circ + r\sqrt{3}L\cos\Theta^\circ + \cdots ,$$

and

$$\int_0^L \{\sin\Theta^\circ - \sqrt{3}s\cos\Theta^\circ + \cdots\}\, ds = r\cos\Theta^\circ + r\sqrt{3}L\sin\Theta^\circ + \cdots . \qquad (20)$$

If the coefficients of the various powers of $\sqrt{3}$ are equated, the equations reduce to

$$\int_0^L \cos\Theta^\circ\, ds = \delta - r\sin\Theta^\circ ,$$

$$\int_0^L \sin\Theta^\circ\, ds = r\cos\Theta^\circ ,$$

$$-\int_0^L \cos\Theta^\circ s\, ds = rL\sin\Theta^\circ ; \qquad (21)$$

and

$$\int_0^L \sin\Theta^\circ s\, ds = rL\cos\Theta^\circ . \qquad (22)$$

It is obvious that the first two equations are satisfied by the initial conditions ($\beta = 0$).

To investigate the significance of the remaining two equations, it becomes necessary to take

$$ds \sin\theta_0 = dx \tag{23}$$

and

$$ds \cos\theta_0 = -dy .$$

With

$$d(sy) = y\,ds + s\,dy ,$$

and

$$d(sx) = x\,ds + s\,dx , \tag{24}$$

the equations (22) become

$$\int_0^{\bar{L}} \cos\theta_0\, s\,ds = \int_0^{\bar{L}} s\,dy = -\left\{sy\right\}_0^{\bar{L}} + \int_0^{\bar{L}} y\,ds ,$$

and

$$\int_0^{\bar{L}} \sin\theta_0\, s\,ds = \int_0^{\bar{L}} s\,dx = \left\{sx\right\}_0^{\bar{L}} - \int_0^{\bar{L}} x\,ds . \tag{25}$$

Whence,

$$\left\{sx\right\}_0^{\bar{L}} = \bar{L}x_0 = \bar{L}r\cos\theta_0 ,$$

and

$$\left\{sy\right\}_0^{\bar{L}} = \bar{L}y_0 = \bar{L}r\sin\theta_0 . \tag{26}$$

It is obvious that the first two equations are satisfied by the initial conditions ($\lambda = 0$).

To investigate the significance of the remaining two equations, it becomes necessary to take

$$dx \sin\theta_0 = dx \tag{23}$$

and

$$ds \cos\theta_0 = -d\lambda .$$

With

$$d(s\lambda) = \lambda\, ds + s\, d\lambda ,$$

and

$$d(sx) = x\, ds + s\, dx , \tag{24}$$

the equations (22) become

$$\int_0^{\bar{\mu}} \cos\theta\, ds = \int_0^{\bar{\mu}} s\, d\lambda = -\left\{s\lambda\right\}_0^{\bar{\mu}} + \int_0^{\bar{\mu}} \lambda\, ds ,$$

and

$$\int_0^{\bar{\mu}} \sin\theta_0\, ds = \int_0^{\bar{\mu}} dx = \left\{sx\right\}_0^{\bar{\mu}} - \int_0^{\bar{\mu}} x\, ds . \tag{25}$$

Hence,

$$\left\{sx\right\}_0^{\bar{\mu}} = \bar{L}\, x_0 = \bar{L}\, r \cos\theta_0 ,$$

and

$$\left\{s\lambda\right\}_0^{\bar{\mu}} = \bar{L}\lambda_0 = \bar{L}\, r \sin\theta_0 . \tag{26}$$

With $\{sx\}_0^{\bar{L}}$ and $\{sy\}_0^{\bar{L}}$ substituted in (25), the identification of the equations shows

$$\int_0^{\bar{L}} \cos\Theta_0 \, s\,ds = -r\bar{L}\sin\Theta_0 = -\bar{L}r\sin\Theta_0 + \int_0^{\bar{L}} y\,ds,$$

or

$$r\bar{L}\sin\Theta_0 + \bar{L}r\sin\Theta_0 = \int_0^{\bar{L}} y\,ds = \eta\bar{L} = 0, \tag{27}$$

and

$$\int_0^{\bar{L}} \sin\Theta_0 \, s\,ds = r\bar{L}\cos\Theta_0 = r\bar{L}\sin\Theta_0 - \int_0^{\bar{L}} x\,ds,$$

or

$$r\bar{L}\cos\Theta_0 - r\bar{L}\sin\Theta_0 = -\int_0^{\bar{L}} x\,ds = -\bar{L}\xi = 0. \tag{28}$$

Here ξ and η are the coordinates of the center of gravity of the spring.

The conditions proposed signify that the center of gravity of the spring coincide with the point O. This can be satisfied only if the rotation $\beta\bar{L}$ of the radius r is very small.

With $\{s x\}_L^0$ and $\{s y\}_L^0$ substituted in (25), the identification of the equations shows

$$\int_0^L \cos\Theta_0 \, s\,ds = -r\bar{L}\sin\Theta_0 = -\bar{L}\, r\sin\Theta_0 + \int_0^{\bar{L}} y\,ds\,,$$

or

$$r\bar{L}\sin\Theta_0 + \bar{L}\, r\sin\Theta_0 = \int_0^{\bar{L}} y\,ds = \eta\bar{L} = 0\,, \tag{27}$$

and

$$\int_0^L \sin\Theta_0 \, s\,ds = r\bar{L}\cos\Theta_0 = r\bar{L}\sin\Theta_0 - \int_0^{\bar{L}} x\,ds\,,$$

or

$$r\bar{L}\cos\Theta_0 - r\bar{L}\sin\Theta_0 = -\int_0^{\bar{L}} x\,ds = -\bar{L}\xi = 0\,. \tag{28}$$

Here ξ and η are the coordinates of the center of gravity of the spring.

The conditions proposed signify that the center of gravity of the spring coincide with the point 0. This can be satisfied only if the rotation βL of the radius r is very small.

3. Second Problem by Phillips

In the first consideration of the problem, it was assumed that r, the radius of the balance wheel, was constant. The problem*(1) can again be considered using r as a variable, and that it varies according to the law $\left(\frac{1}{r} - \frac{1}{r_0}\right) = \beta$ = constant.

This formula can be written as

$$r_0 - r = \beta r_0 r$$

from which comes

$$r = \frac{r_0}{1+\beta r_0} = r_0 - \beta r_0^2 . \qquad (29)$$

If β is assumed small, (i.e. the reaction force), the conditional equations (16) still hold, and the expansion $r = r_0 - \beta r_0^2$ substituted for r gives

$$\int_0^{\bar{L}} \{\cos\theta_0 + \beta s \sin\theta_0 + \cdots\} ds = \delta - (r_0 - \beta r_0^2)\{\sin\Phi_0 - \beta L \cos\Phi_0 + \cdots\},$$

and (30)

$$\int_0^{\bar{L}} \{\sin\theta_0 - \beta s \cos\theta_0 + \cdots\} ds = (r_0 - \beta r_0^2)\{\cos\Phi_0 - \beta L \sin\Phi_0 + \cdots\}.$$

From these results

$$\begin{aligned} \int_0^{\bar{L}} \cos\theta_0 ds &= \delta - r_0 \sin\Phi_0 , \\ \int_0^{\bar{L}} \sin\theta_0 ds &= r_0 \cos\Phi_0 , \\ \int_0^{\bar{L}} \sin\theta_0 s ds &= r_0 L \cos\Phi_0 + r_0^2 \sin\Phi_0 , \end{aligned} \qquad (31)$$

3. Second Problem by Phillips

In the first consideration of the problem, it was assumed that r, the radius of the balance wheel, was constant. The problem*(1) can again be considered using r as a variable, and that it varies according to the law $\left(\frac{1}{r} - \frac{1}{r_0}\right) = \lambda = \text{constant}$.

This formula can be written as

$$r_0 - r = \lambda r_0 r$$

from which comes

$$r = \frac{r_0}{1+\lambda r_0} = r_0 - \lambda r_0^2 . \qquad (29)$$

If λ is assumed small, (i.e. the reaction force), the conditional equations (16) still hold, and the expression $r = r_0 - \lambda r_0^2$ substituted for r gives

$$\int_0^L \{\cos\theta^0 + \lambda 3 \sin\theta^0 + \cdots\} dz = \{2 - (r_0 - \lambda r_0^2)\}\{\sin\theta^0 - \lambda 4 \cos\theta^0 + \cdots\}$$

and

$$\int_0^{\bar{\mu}} \{\sin\theta^0 - \lambda 3 \cos\theta^0 + \cdots\} dz = (r_0 - \lambda r_0^2)\{\cos\theta^0 - \lambda 3 \sin\theta^0 + \cdots\} \qquad (30)$$

From these results

$$\int_0^{\bar{\mu}} \sin\theta^0 dz = r_0 \bar{\mu} \cos\theta^0 + r_0^2 \sin\theta^0 ,$$

$$\int_0^{\bar{\mu}} \sin\theta^0 dz = r_0 \cos\theta^0 ,$$

$$\int_0^{L} \cos\theta^0 dz = 2 - r_0 \sin\theta^0 . \qquad (31)$$

and

$$-\int_0^{\bar{L}} \cos\theta_0 \, s \, ds = r_0 \bar{L} \sin\Theta_0 - r_0^2 \cos\Theta_0 . \qquad (32)$$

Integration by parts yields

$$\int_0^{\bar{L}} \cos\theta_0 \, s \, ds = -\{ys\}_0^{\bar{L}} + \int_0^{\bar{L}} y \, ds = -\left\{ r_0 \bar{L} \sin\Theta_0 - r_0^2 \cos\Theta_0 \right\},$$

and

$$\int_0^{\bar{L}} \sin\theta_0 \, s \, ds = \{xs\}_0^{\bar{L}} - \int_0^{\bar{L}} x \, ds = \left\{ r_0 \bar{L} \cos\Theta_0 + r_0^2 \sin\Theta_0 \right\}; \qquad (33)$$

for which

$$\{ys\}_0^{\bar{L}} = y_0 \bar{L} = r_0 \bar{L} \sin\Theta_0 \qquad (34)$$

and

$$\{xs\}_0^{\bar{L}} = x_0 \bar{L} = r_0 \bar{L} \cos\Theta_0 ,$$

also,

$$\bar{L}\eta = \int_0^{L} y \, ds = -r_0 \bar{L} \sin\Theta_0 + r_0^2 \cos\Theta_0 + r_0 \bar{L} \sin\Theta_0 , \qquad (35)$$

or $$\bar{L}\eta = r_0^2 \cos\Theta_0 ,$$

and

$$-\bar{L}\xi = -\int_0^{\bar{L}} x \, ds = r_0 \bar{L} \cos\Theta_0 + r_0^2 \sin\Theta_0 - r_0 \bar{L} \cos\Theta_0 , \qquad (36)$$

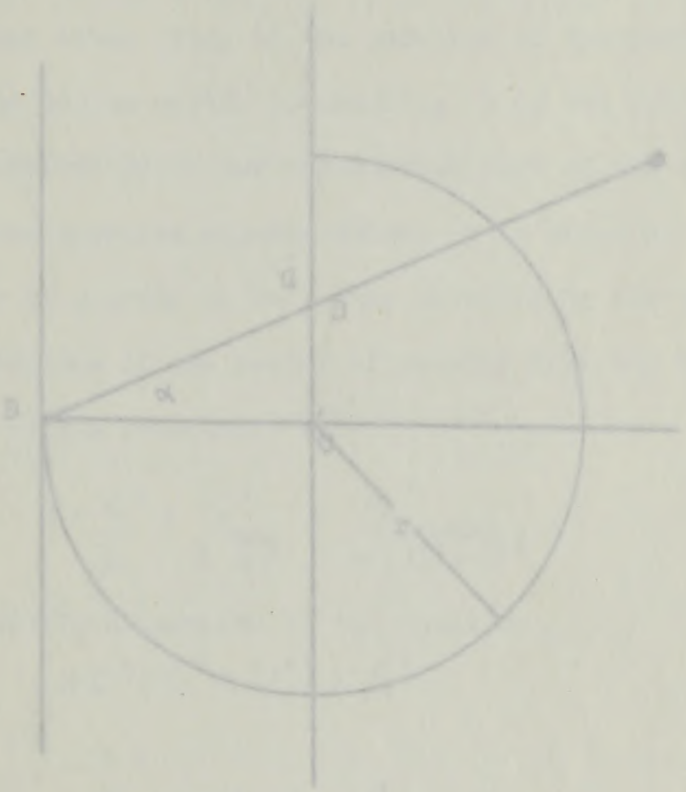
or $$-\bar{L}\xi = r_0^2 \sin\Theta_0 .$$

Further,

$$\sqrt{\xi^2 + \eta^2} = D = \frac{r_0^2}{\bar{L}} , \text{ and } -\frac{\eta}{\xi} = \cot\Theta_0 . \qquad (37)$$

The center of gravity G of the spring is now on the line OG parallel to the first element B of the circumference in its initial state, or it lies on a perpendicular to the line OB. The distance OG = D satisfies the relation

$$DL = r_0^2 \ .$$



The center of gravity O of the spring is now on the line OO parallel to the first element B of the circumference in its initial state, or it lies on a perpendicular to the line OE. The distance OO = D satisfies the relation

$$DL = r_0^2 .$$

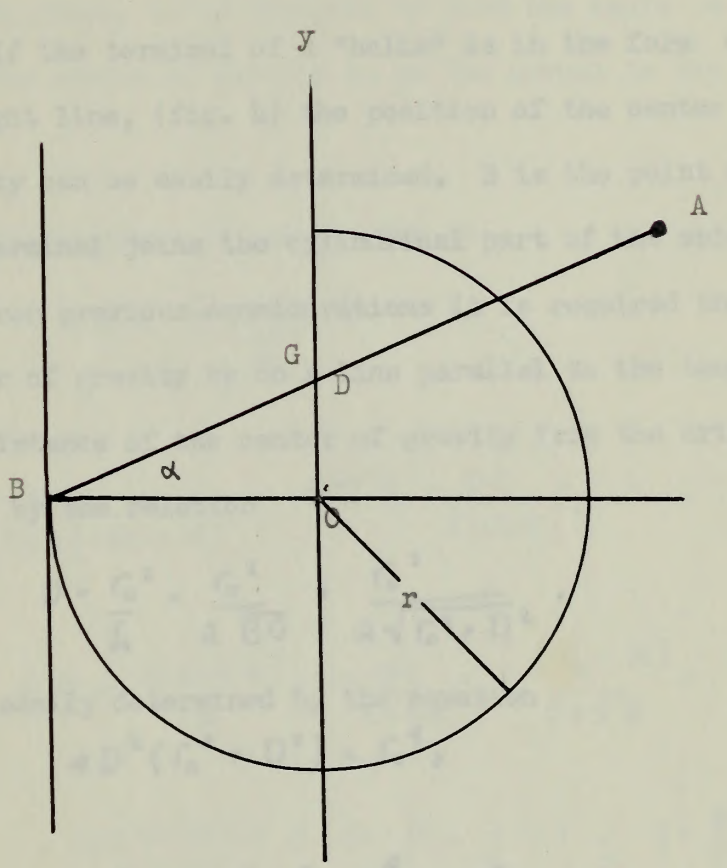
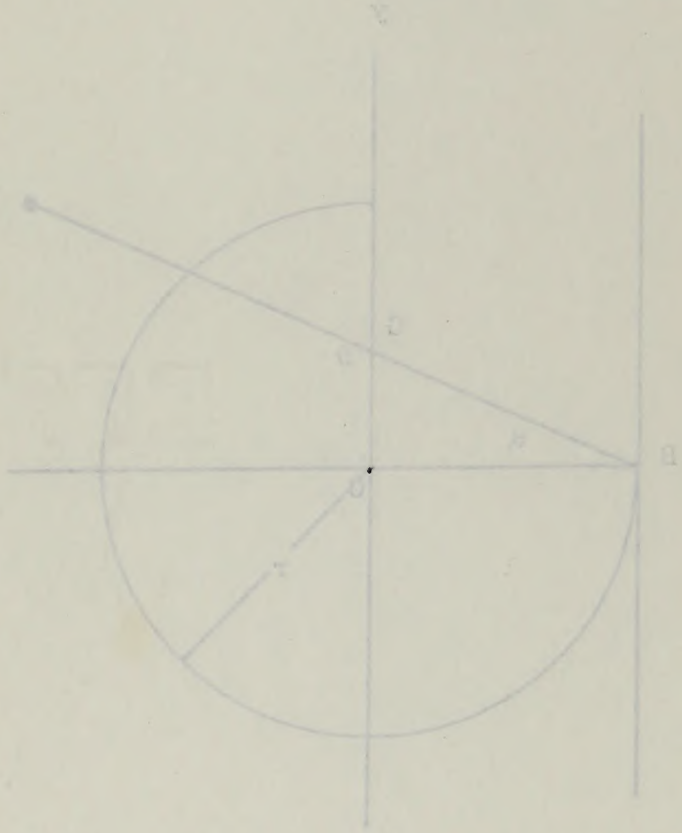


Figure 4. Straight Line Terminal


A
B
C
O

4. Straight Line Terminals

The second problem by Phillips permits the designing of the fastened terminals of a helix such that the reaction force at the end points is small.

If the terminal of a "helix" is in the form of a straight line, (fig. 4) the position of the center of gravity can be easily determined. B is the point where the terminal joins the cylindrical part of the spiral, and from previous considerations it is required that the center of gravity be on a line parallel to the tangent at B. The distance of the center of gravity from the origin is given by the relation

$$D = \frac{r_o^2}{\bar{l}_4} = \frac{r_o^2}{2\,\overline{BG}} = \frac{r_o^2}{2\sqrt{r_o^2 + D^2}} . \qquad (38)$$

D is easily determined by the equation

$$4D^2(r_o^2 + D^2) = r_o^4 ,$$

or

$$4D^4 + 4r_o^2 D^2 - r_o^4 = 0 ;$$

where,

$$D^2 = 0.202\, r_o^2 ; \quad D = 0.045\, r_o$$

and

$$\alpha = 24^\circ 28' .$$

4. Straight Line Terminals

The second problem by Phillips permits the designing of the fastened terminals of a helix such that the reaction force at the end points is small.

If the terminal of a "helix" is in the form of a straight line, (fig. 4) the position of the center of gravity can be easily determined. B is the point where the terminal joins the cylindrical part of the spiral, and from previous considerations it is required that the center of gravity be on a line parallel to the tangent at B. The distance of the center of gravity from the origin is given by the relation

$$D = r_0 \frac{K}{\alpha} = r_0 \frac{r_0}{\alpha \beta G} = \frac{r_0^2}{\alpha \sqrt{r_0^2 + D^2}} \quad (30)$$

D is easily determined by the equation

$$4D^2(r_0^2 + D^2) = r_0^4,$$

or

$$4D^4 + 4r_0^2 D^2 - r_0^4 = 0 ;$$

where

$$D^2 = 0.2062\, r_0^2; \quad D = 0.4542\, r_0$$

and

$$\alpha = 24^\circ 28'.$$

5. One Arc of a Circle

The center C of the arc BDE of radius R is located at a distance ($r_0 - R$) from the origin O, (fig. 5). The terminal arc joins the cylindrical part of the spring at B. R being given, it is required to find the angle α such that the center of gravity is on the normal to the radius OB.

The centroid of the arc BDE is

$$CG = R\frac{\sin\alpha}{\alpha};$$

however,

$$\frac{OC}{\cos(180-\alpha)} = CG = -\frac{OC}{\cos\alpha}.$$

Now,

$$CG = R\frac{\sin\alpha}{\alpha} = -\frac{OC}{\cos\alpha} = -\frac{(r_0 - R)}{\cos\alpha}, \qquad (39)$$

and

$$-\frac{\sin\alpha\cos\alpha}{\alpha} = \frac{r_0 - R}{R} = \frac{1}{\rho} - 1 \; ; \; \rho = \frac{R}{r_0}. \qquad (39a)$$

Phillips' condition is:

$$\overline{OG} = D = \frac{r_0^2}{L},$$

but for the arc of a circle

$$D = GC\sin\alpha = R\frac{\sin\alpha}{\alpha}\sin\alpha, \qquad (40)$$

where the arc length $L = 2\alpha R$.

5. The Arc of a Circle

The center C of the arc DGE of radius R is located at a distance $(r_o - R)$ from the origin O. (Fig. 5). The terminal arc joins the cylindrical part of the spring at D. R being given, it is required to find the angle α such that the center of gravity is on the normal to the radius OE.

The centroid of the arc DGE is

$$CG = R \frac{\sin\alpha}{\alpha} ;$$

however:

$$\frac{OC}{\cos(180 - \alpha)} = CG = - \frac{OC}{\cos\alpha} .$$

Now:

$$CG = R \frac{\sin\alpha}{\alpha} = - \frac{OC}{\cos\alpha} = - \frac{(r_o - R)}{\cos\alpha} , \qquad (39)$$

and

$$- \frac{\sin\alpha \cos\alpha}{\alpha} = \frac{r_o - R}{R} = \frac{\delta}{L} - 1 ; \quad \delta = \frac{R}{r_o} . \qquad (39a)$$

Phillips' condition is:

$$\frac{OG}{} = D = \frac{r_o^2}{R} ,$$

but for the arc of a circle

$$D = GC \sin\alpha = R \frac{\sin\alpha}{\alpha} \sin\alpha ; \qquad (40)$$

where the arc length $L = 2\alpha R$.

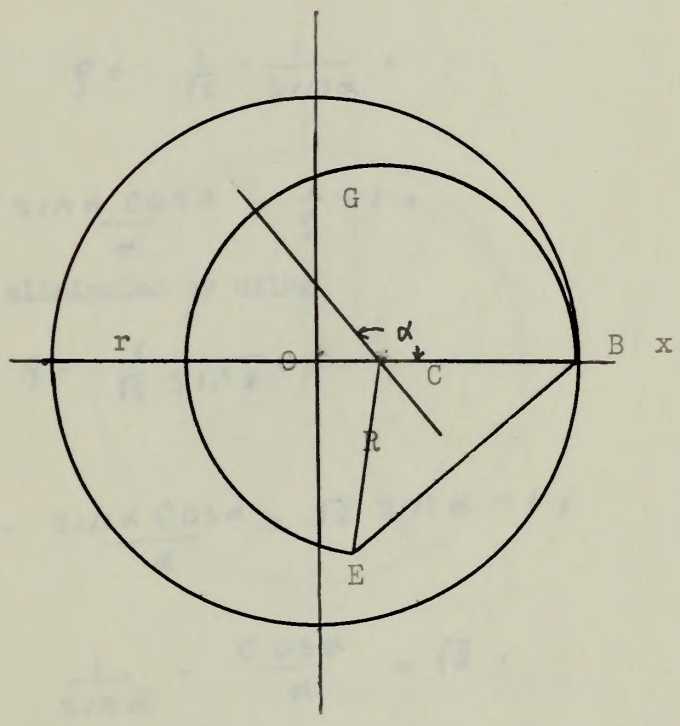
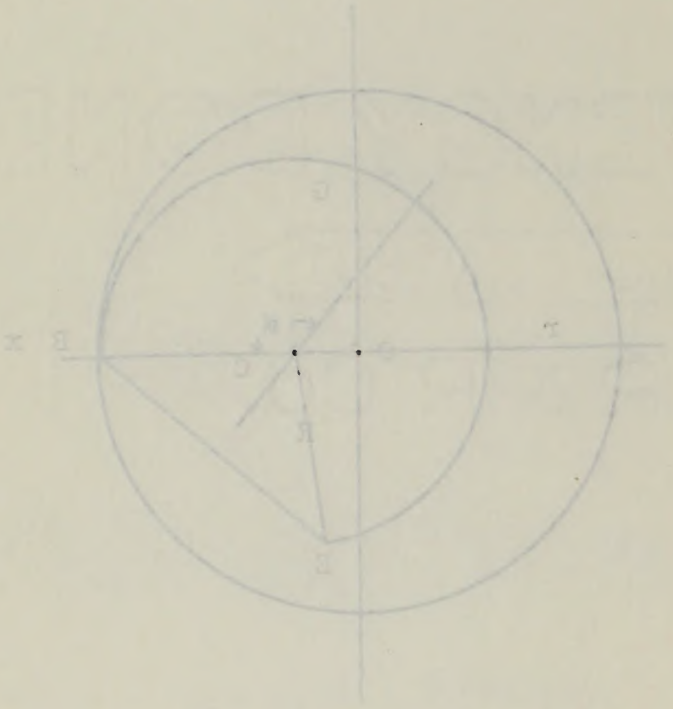


FIGURE 5. ONE ARC OF A CIRCLE



Therefore,

$$R \frac{\sin^2\alpha}{\alpha} I_4 = r_o^2 \; ; \; 2\alpha R^2 \frac{\sin^2\alpha}{\alpha} = r_o^2 , \tag{40a}$$

$$2 \sin^2\alpha = \frac{r_o^2}{R^2} = \rho^2 , \tag{41}$$

and

$$\rho = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sin\alpha} . \tag{41a}$$

From

$$- \frac{\sin\alpha \cos\alpha}{\alpha} = \frac{1}{\rho} - 1 ,$$

ρ is eliminated by using

$$\rho = \frac{1}{\sqrt{2} \sin\alpha} ,$$

hence,

$$- \frac{\sin\alpha \cos\alpha}{\alpha} = \sqrt{2} \sin\alpha - 1 ,$$

or

$$\frac{1}{\sin\alpha} - \frac{\cos\alpha}{\alpha} = \sqrt{2} . \tag{42}$$

The value of α is found by trial substitution. H. Bouasse*(1) found that $\alpha = 121^\circ 13'$, $\sin\alpha = .0855$, and $R = .0827 r_o$.

Therefore:

$$K \frac{\sin^2\alpha}{\alpha} L = r_{o_s}^2 \quad \text{or} \quad \alpha \frac{K \sin^2\alpha}{\alpha} = r_{o_s}^2 \tag{10a}$$

$$\alpha \sin^2\alpha = \frac{r_{o_s}}{R_s} = \rho_s^2 \tag{11}$$

and

$$\rho = \frac{r_s}{l} \cdot \frac{1}{\sin\alpha} \tag{11a}$$

From

$$\frac{\sin\alpha \cos\alpha}{\alpha} = \frac{s}{l} - 1 ,$$

ρ is eliminated by using

$$\rho = \frac{r_s}{l} \frac{1}{\sin\alpha} ,$$

Hence:

$$\frac{\sin\alpha \cos\alpha}{\alpha} = r_s \sin\alpha - 1 ,$$

or

$$\frac{\sin\alpha}{1} - \frac{\cos\alpha}{\alpha} = r_s . \tag{12}$$

The value of α is found by trial substitution.

H. Dousse*(1) found that $\alpha = 121°13'$, $\sin\alpha = .0855$,

and $R = .0327r_o$.

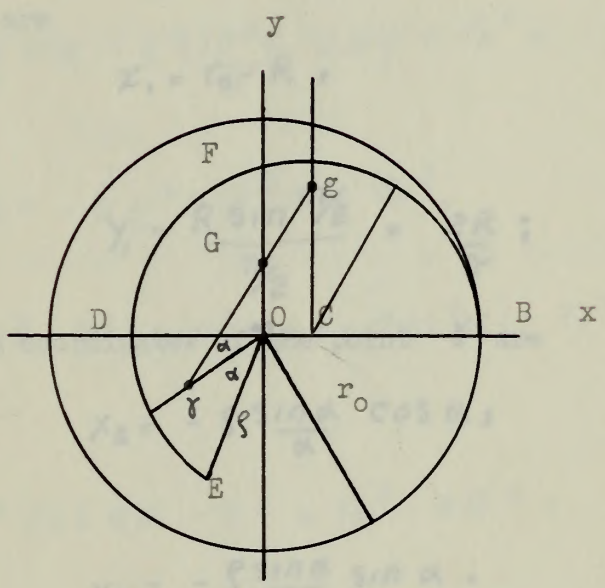
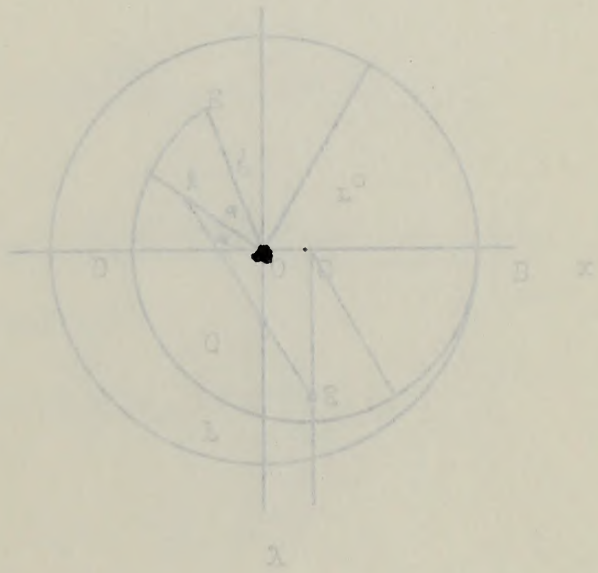


FIGURE 6. TWO ARCS OF A CIRCLE



6. Two Arcs of a Circle

The first arc BFD has a radius of curvature R and center c (fig. 6). The second arc DE has the center at O, makes an angle 2α with the x axis, and has a radius of curvature $\rho = 2R - r_0$.

The coordinates*(1) of the point g, the centroid of the arc, are

$$x_1 = r_0 - R, \tag{43}$$

and

$$y_1 = \frac{R \sin \pi/2}{\pi/2} = \frac{2R}{\pi}; \tag{44}$$

also, the coordinates of the point γ are

$$x_2 = -\rho \frac{\sin\alpha}{\alpha} \cos\alpha, \tag{45}$$

and

$$y_2 = -\rho \frac{\sin\alpha}{\alpha} \sin\alpha. \tag{46}$$

If the moments along the x axis are equated ($x_1 l_1 = x_2 l_2$), the resulting equation is

$$(r_0 - R) R\pi = \rho \frac{\sin\alpha \cos\alpha}{\alpha} 2\alpha\rho,$$

or

$$(r_0 - R) R\pi = (2R - r_0)^2 \sin^2\alpha. \tag{47}$$

6. Two Arcs of a Circle

The first arc EFD has a radius of curvature R and center c (fig. 6). The second arc DE has the center at O, makes an angle 2α with the x axis, and has a radius of curvature $\rho = \alpha R - r_o$.

The coordinates*(1) of the point E, the centroid of the arc, are

$$x_1' = r_o - R \, , \tag{43}$$

and

$$y_1' = \frac{R \sin \frac{\pi}{\alpha}}{\frac{\pi}{2\alpha}} = \frac{2R}{\pi} \, ; \tag{44}$$

also, the coordinates of the point ξ are

$$x_2' = -\rho \frac{\sin\alpha}{\alpha} \cos\alpha \, , \tag{45}$$

and

$$y_2' = -\rho \frac{\sin\alpha}{\alpha} \sin\alpha \, . \tag{46}$$

If the moments along the x axis are equated ($x_1' f_1' = x_2' f_2'$), the resulting equation is

$$(r_o - R) R \pi = \rho \frac{\sin\alpha \cos\alpha}{\alpha} 2\alpha \rho \, ,$$

or

$$(r_o - R) R \pi = (\alpha R - r_o)^2 \sin^2 \alpha \, . \tag{47}$$

The distance $\overline{OG}$ of the center of gravity of the system must satisfy the conditions of Phillips,

$$DL = r_0^2.$$

Hence, it can be written that

$$y_1 l_1 - y_2 l_2 = r_0^2,$$

or

$$\frac{2R}{\pi} \pi R - \rho \frac{\sin^2\alpha}{\alpha} 2\alpha\rho = r_0^2,$$

$$2R^2 - 2\rho^2 \sin^2\alpha = r_0^2, \tag{48}$$

$$2R^2 - \rho^2 [\sin^2\alpha + 1 - \cos^2\alpha] = r_0^2,$$

or

$$\rho^2 \cos 2\alpha - \rho^2 = r_0^2 - 2R^2,$$

or

$$(2R - r_0)^2 \cos 2\alpha = r_0^2 - 2R^2 + (2R - r_0)^2,$$

$$(2R - r_0)^2 \cos 2\alpha = 2(r_0 - R)^2. \tag{49}$$

With $\frac{R}{r_0} = \beta$, and the addition of equation (47) and (49), term by term gives

$$r_0^2(2\beta - 1)^2 = 2r_0^2(1-\beta)^2 + r_0(1-\beta)R\pi,$$

The distance $\overline{OG}$ of the center of gravity of the system must satisfy the conditions of Phillips.

$$\overline{DL} = r_o .$$

Hence, it can be written that

$$\lambda' \bar{r}_1 - \lambda'' \bar{r}_2 = \bar{r}_o ,$$

or

$$\frac{m}{2R} \overline{m_1 R} - \frac{m}{2R} \overline{m_2 \alpha} \; \overline{\alpha \alpha} = \bar{r}_o ,$$

$$a R_s - a \rho_s \sin^2 \alpha = r_o , \tag{48}$$

$$a R_s - a \rho_s [\sin^2 \alpha + 1 - \cos^2 \alpha] = r_o ,$$

or

$$\rho_s \cos 2\alpha - \rho_s = r_o - 2R ,$$

or

$$(2R - r_o)_s \cos 2\alpha = r_o - 2R_s + (2R - r_o)_s ,$$

$$(2R - r_o)_s \cos 2\alpha = \lambda (r_o - R)_s . \tag{49}$$

With $\frac{R}{r_o} = \lambda$, and the addition of equation (47) and (49), term by term gives

$$r_o^2 (2\lambda - 1)_s = 2 r_o^2 (1 - \lambda)^2 + r_o^2 (1 - \lambda) R_m ,$$

and both sides squared gives

$$(2\beta-1)^4 = \pi^2\beta^2(1-\beta)^2 + 4(1-\beta)^4,$$

or

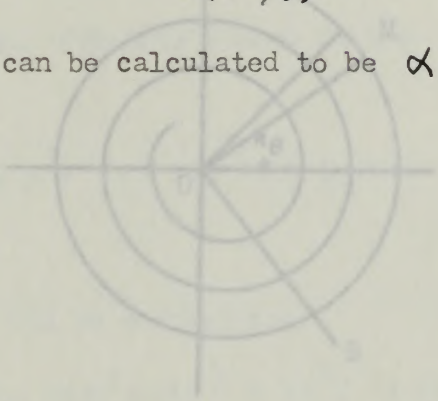
$$(\pi^2-12)\beta^4 + (16-2\pi^2)\beta^3 + \pi^2\beta^2 - 8\beta + 3 = 0.$$

One root of this equation is $\beta^{*(1)} = .0832$.

Equation (47) divided by (49), term by term, gives

$$\tan 2\alpha = \frac{\pi\beta}{2(1-\beta)}. \qquad (50)$$

The angle α can be calculated to be $\alpha = 41^\circ 21'$ *(1).



and both sides squared gives

$$(2\sqrt{3}-1)^4 = \Pi^2_s \sqrt{3}_s (1-\sqrt{3})^2_s + 4\,(1-\sqrt{3})^4 .$$

or

$$(11\sqrt{3}-12)\sqrt{3}_4 + (16 - 9\Pi^2_s)\sqrt{3}_3 + \Pi^2_s \sqrt{3}^2_s - 8\sqrt{3} + 3 = 0 .$$

One root of this equation is $\sqrt{3}_{*(I)} = .0832$.

Equation (47) divided by (48), term by term, gives

$$\text{Tan } 2\alpha = \frac{\Pi\sqrt{3}}{9(1-\sqrt{3})} . \qquad (50)$$

The angle α can be calculated to be $\alpha = 11°21', {}_{*(I)}$.

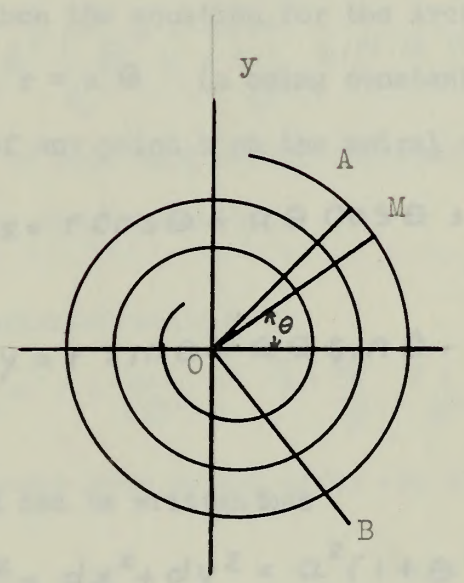


FIGURE 7. ARCHIMEDES SPIRAL

7. Archimedes Spiral

Phillips' first problem is satisfied at least approximately by a curve having its center of gravity on the axis of the balance wheel. Now it is required to design a spring, having the shape of an Archimedes*(1) spiral, such that the center of gravity lies at the pole (fig. 7). Since the equation for the Archimedes spiral is given by $r = a\theta$ (a being constant), the coordinates of any point M on the spiral are

$$x = r\cos\theta = a\theta\cos\theta ,$$

and

$$y = r\sin\theta = a\theta\sin\theta . \tag{51}$$

Therefore, it can be written that

$$ds^2 = dx^2 + dy^2 = a^2(1+\theta^2)\,d\theta^2 , \tag{52}$$

or

$$ds = a\sqrt{(1+\theta^2)}\,d\theta . \tag{53}$$

The coordinates of the center of gravity of the spiral are given by the equations

$$M\xi = \int x\,ds = \int_{\theta_0}^{\theta} a^2\theta\cos\theta(1+\theta^2)^{1/2}\,d\theta , \tag{53a}$$

7. Archimedes Spiral

Phillips' first problem is satisfied at least approximately by a curve having the center of gravity on the axis of the balance wheel. Now it is required to design a spring, having the shape of an Archimedes*(1) spiral, such that the center of gravity lies at the pole (Fig. 7). Since the equation for the Archimedes spiral is given by $r = a\theta$ (a being constant), the coordinates of any point M on the spiral are

$$x = r\cos\theta = a\theta\cos\theta ,$$

and

$$y = r\sin\theta = a\theta\sin\theta .$$

(31)

Therefore, it can be written that

$$ds_s^2 = dx_s^2 + dy_s^2 = a_s^2(1+\theta_s)\,d\theta_s^2 , \quad (32)$$

or

$$ds = a\sqrt{(1+\theta_s)}\,d\theta . \quad (33)$$

The coordinates of the center of gravity of the spiral are given by the equations

$$\bar{x}\frac{s}{2} = \int_{\theta_0}^{\theta} x\,ds = \int_{\theta_0}^{\theta} a_s\theta\cos\theta(1+\theta_s)^{1/2}\,d\theta , \quad (33a)$$

and

$$L\eta = \int y\,ds = \int_{\theta_0}^{\theta} a^2\theta \sin\theta\,(1+\theta^2)^{1/2}\,d\theta, \qquad (53b)$$

where L is the length of the spiral utilized.

The equations can also be written as

$$L\xi = a^2\int_{\theta_0}^{\theta}\theta^2\sqrt{1+\frac{1}{\theta^2}}\cos\theta\,d\theta,$$

and (54)

$$L\eta = a^2\int_{\theta_0}^{\theta}\theta^2\sqrt{1+\frac{1}{\theta^2}}\sin\theta\,d\theta;$$

also, it can be seen that except for small values of θ the radical is close to unity.

In the expanded radical

$$\sqrt{1+\frac{1}{\theta^2}} = 1+\frac{1}{2\theta^2}+\frac{1/2\,(1/2-1)}{2!}\frac{1}{\theta^4}+\cdots, \qquad (55)$$

the terms greater than second order are neglected.

It follows that

$$a^2\int_{\theta_0}^{\theta}\theta^2\cos\theta\,d\theta+\int_{\theta_0}^{\theta}\frac{a^2}{2}\cos\theta\,d\theta = a^2\left[2\theta\cos\theta+(\theta^2-2)\sin\theta + 1/2\sin\theta\right]_{\theta_0}^{\theta}, \qquad (55a)$$

or

$$L\xi = a^2\left[\theta^2\sin\theta+2\theta\cos\theta-3/2\sin\theta\right]_{\theta_0}^{\theta};$$

and that

$$a^2\int_{\theta_0}^{\theta}\theta^2\sin\theta\,d\theta+\int_{\theta_0}^{\theta}\frac{a^2}{2}\sin\theta\,d\theta = a^2\left[2\theta\sin\theta-(\theta^2-2)\cos\theta - \tfrac{1}{2}\cos\theta\right]_{\theta_0}^{\theta},$$

and

$$L_1 = \int_{\theta^\circ}^{\theta} a_s \Theta \sin\theta\,(1+\epsilon_s)\sqrt{\epsilon}\,d\theta, \qquad (23b)$$

where L_1 is the length of the spiral utilized.

The equations can also be written as

$$L_2 = a_s \int_{\theta^\circ}^{\theta} \Theta_s \sqrt{1+\frac{1}{\Theta_s^2}}\,\cos\theta\,d\theta,$$

and (24)

$$L_1 = a_s \int_{\theta^\circ}^{\theta} \Theta_s \sqrt{1+\frac{1}{\Theta_s^2}}\,\sin\theta\,d\theta;$$

also, it can be seen that except for small values of Θ the radical is close to unity.

In the expanded radical

$$\sqrt{1+\frac{1}{\Theta_s^2}} = 1 + \frac{1}{2\Theta_s^2} + \frac{3}{2!}\frac{(\sqrt{\epsilon}-1)}{\Theta_s^4} + \cdots \qquad (25)$$

the terms greater than second order are neglected.

It follows that

$$a_s \int_{\theta^\circ}^{\theta} \Theta_s \cos\theta\,d\theta + \int_{\theta^\circ}^{\theta} \frac{a_s}{2}\cos\theta\,d\theta = a_s\left[\theta\cos\theta + (\theta-1)\sin\theta + \sqrt{\epsilon}\sin\theta\right]_{\theta^\circ}^{\theta}, \qquad (25a)$$

or

$$L_2 = a_s\left[\Theta_s\sin\theta + 2\theta\cos\theta - 3\sqrt{\epsilon}\sin\theta\right]_{\theta^\circ}^{\theta},$$

and that

$$a_s \int_{\theta^\circ}^{\theta} \Theta_s \sin\theta\,d\theta + \int_{\theta^\circ}^{\theta} \frac{a_s}{2}\sin\theta\,d\theta = a_s\left[-2\theta\sin\theta - (\theta-1)\cos\theta\right.$$

$$\left. - \frac{1}{2}\cos\theta\right]_{\theta^\circ}^{\theta},$$

or

$$L\,\eta = a^2\left[-\theta^2\cos\theta + 2\theta\sin\theta + \tfrac{3}{2}\cos\theta\right]_{\theta_0}^{\theta}. \quad (55b)$$

For a spring having many spires, "a" is small compared to the radius vector r which corresponds to the extremity A of the spring utilized.

The value of the equations (55a) and (55b) evaluated at θ_0, are very small; hence, can be neglected. Except for θ^2, the terms in the equations (55a) and (55b) evaluated at the extremity A can also be neglected.

A fairly precise value for the coordinates of the center of gravity is:

$$L\,\xi = r^2\sin\theta \quad (56)$$

or

$$L\,\eta = -r^2\cos\theta\,; \quad (56a)$$

however,

$$L\sqrt{\xi^2+\eta^2} = r^2\,; \quad D = \sqrt{\xi^2+\eta^2} = \frac{r^2}{L}, \quad (57)$$

and

$$\frac{\eta}{\xi} = -\frac{r^2\cos\theta}{r^2\sin\theta} = -\cot\theta\,.$$

These results signify that the center of gravity g of the spiral is on the line OB, normal to the line OA, going from the pole O to the extremity A of the spiral,

or

$$L\,\eta = a_s\left[-\theta^2\cos\theta + 2\,\theta\,\sin\theta + 2\cos\theta\right]_{\theta_0}^{\theta}. \qquad (55b)$$

For a spring having many spires, "a" is small compared to the radius vector r which corresponds to the extremity A of the spring utilized.

The value of the equations (55a) and (55b) evaluated at θ_0, are very small; hence, can be neglected. Except for θ_s, the terms in the equations (55a) and (55b) evaluated at the extremity A can also be neglected.

A fairly precise value for the coordinates of the center-of gravity is:

$$L\,\xi = r_s^2 \sin\theta \qquad (56)$$

or

$$L\,\eta = -r_s^2\cos\theta\,; \qquad (56a)$$

however:

$$L\sqrt{\xi_s^2+\eta_s^2} = r_s^2\,; \quad D = \sqrt{\xi_s^2+\eta_s^2} = \frac{r_s^2}{L}\,. \qquad (57)$$

and

$$\frac{\eta}{\xi} = -\frac{r_s^2\cos\theta}{r_s^2\sin\theta} = -\cot\theta\,.$$

These results signify that the center of gravity G of the spiral is on the line OG, normal to the line OA, going from the pole O to the extremity A of the spiral.

and is located on the same side of the origin as the terminal of the spring.

An approximate length for the spiral is $\bar{L} = \frac{a\theta^2}{2}$. This value of $\bar{L}$ substituted in equation (57) gives that

$$D = a^2\theta^2 / a\frac{\theta^2}{2} = 2a.$$

This expresses that the distance of the center of gravity from the pole is equal to twice the value of the parameter in the formula $r = a\theta$ defining the spiral.

and is located on the same side of the origin as the terminal of the spring.

An approximate length for the spiral is $\bar{\mu} = \frac{a\theta_s}{2}$. This value of $\bar{\mu}$ substituted in equation (57) gives that

$$\mathcal{T} = a_s\theta_s\sqrt{a\frac{\theta_s}{2}} = \frac{a}{2}\rho .$$

This expresses that the distance of the center of gravity from the pole is equal to twice the value of the parameter in the formula $r = a\theta$ defining the spiral.

8. Method of Resal

Figure 8 represents a spring*(1) of uniform radius ρ_0, having one extremity A fastened to a stud, and the other extremity B fastened to a ferrule which is allowed to rotate about the origin of the coordinate system.

Any distortion of the spring is equivalent to the application of a force F, having components $F_x F_y$, and the moment of forces C to the point A.

The coordinates of the stud and the length of the spring are respectively

$$x = \rho_0 , \quad y = 0 ,$$

and

$$\bar{l} = \rho_0 \Theta .$$

If the ferrule is rotated through an angle Θ, the coordinates of the extremity B become

$$x = \rho_0 \cos (\Theta + \theta) ,$$

and

$$y = \rho_0 \sin (\Theta + \theta) . \qquad (58)$$

xy and ρ are the coordinates and the radius of curvature of an arbitrary point along the spring after deformation, and the direction of the radius of curvature makes an angle φ with the x axis. Any element along

8. Method of Mesal

Figure 8 represents a spring*(1) of uniform radius ρ_o, having one extremity A fastened to a stud, and the other extremity B fastened to a ferrule which is allowed to rotate about the origin of the coordinate system.

Any distortion of the spring is equivalent to the application of a force F, having components $F_x F_y$, and the moment of forces C to the point A.

The coordinates of the stud and the length of the spring are respectively

$$x = \rho_o \, , \quad y = o \, ,$$

and

$$L = \rho_o \Theta .$$

If the ferrule is rotated through an angle Θ, the coordinates of the extremity B become

$$x = \rho_o \cos (\Theta + \varepsilon) \, ,$$

and

$$y = \rho_o \sin (\Theta + \varepsilon) .$$

(58)

xy and ρ are the coordinates and the radius of curvature of an arbitrary point along the spring after deformation, and the direction of the radius of curvature makes an angle ψ with the x axis. Any element along

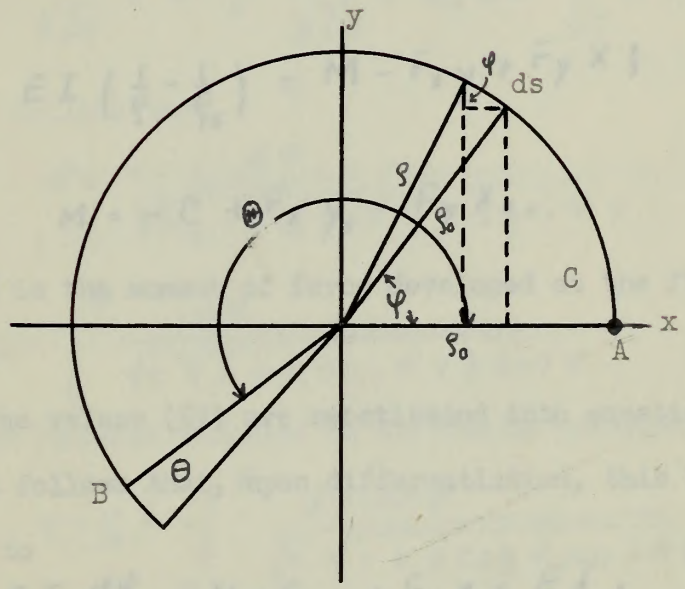
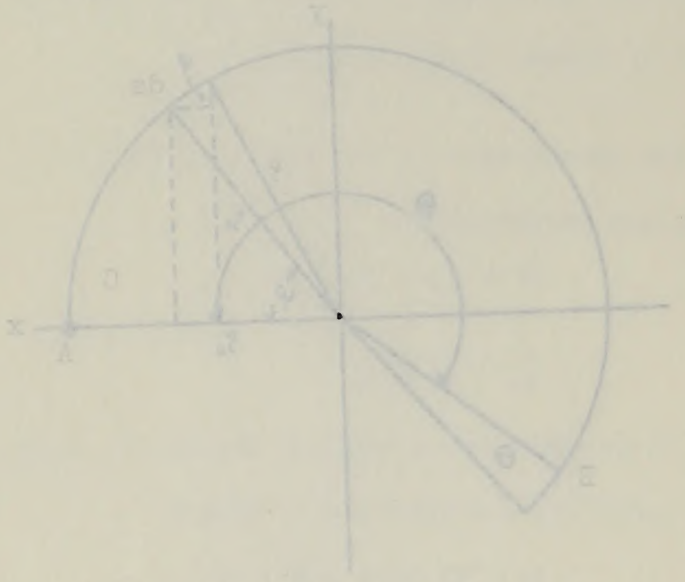


FIGURE 8. "HELICAL" SPRING



the spring can now be represented by

$$\frac{1}{\rho} = \frac{d\varphi}{ds}, \quad \frac{dx}{ds} = -\sin\varphi, \text{ and } \frac{dy}{ds} = \cos\varphi . \tag{59}$$

From previous considerations it was found that the equation for the deformation of the spring is

$$EI\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) = -F_x(y-y_0) + F_y(x-x_0) - C, \tag{60}$$

or

$$EI\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) = M - F_x y + F_y x;$$

where,

$$M = -C + F_x y_0 - F_y x_0 .$$

M is the moment of force developed at the fixed point.

The values (59) are substituted into equation (60) and it follows that, upon differentiation, this equation leads to

$$EI\frac{d\varphi}{ds} = M - F_x y + F_y x + \frac{EI}{\rho_0},$$

or

$$EI\frac{d^2\varphi}{ds^2} = -F_x\frac{dy}{ds} + F_y\frac{dx}{ds},$$

or

$$EI\frac{d^2\varphi}{ds^2} = -F_x\cos\varphi - F_y\sin\varphi . \tag{61}$$

Equation (61) multiplied by $d\varphi$ and integrated shows

$$\frac{1}{2}\left(\frac{d\varphi}{ds}\right)^2 = -\frac{F_x}{EI}\sin\varphi + \frac{F_y}{EI}\cos\varphi + \frac{\delta}{EI},$$

or

$$\frac{d\varphi}{ds} = \sqrt{-\frac{F_x 2C \sin\varphi}{EIC} + \frac{F_y 2C}{EIC}\cos\varphi + \frac{2\delta}{EI}}.$$

The constants can be replaced by

$$-\frac{2F_x}{EIC} = b, \quad \frac{2F_y}{EIC} = a, \text{ and } \frac{2\delta}{EI} = C.$$

It follows that

$$ds = \frac{d\varphi}{\sqrt{C}\sqrt{1 + a\cos\varphi + b\sin\varphi}},$$

or

$$s = \int_0^{\varphi} \frac{d\varphi}{\sqrt{C}\sqrt{1 + a\cos\varphi + b\sin\varphi}}. \tag{62}$$

The equation can also be written as components

$$x - \xi_0 = -\int_0^{\varphi} \frac{d\varphi \sin\varphi}{\sqrt{C}\sqrt{1 + a\cos\varphi + b\sin\varphi}}, \tag{62a}$$

and

$$y = \int_0^{\varphi} \frac{d\varphi\cos\varphi}{\sqrt{C}\sqrt{1 + a\cos\varphi + b\sin\varphi}}. \tag{62b}$$

To integrate these expressions, it is assumed that the quantities a and b are small; hence, they can be expanded into a cosine and sine series:

$$(1 + a\cos\varphi + b\sin\varphi)^{-1/2} = 1 - \frac{a}{2}\cos\varphi - \frac{b}{2}\sin\varphi + \cdots.$$

$$(1+\alpha\cos\psi+\beta\sin\psi)^{-\frac{1}{2}} = 1 - \frac{s}{2}\alpha\cos\psi - \frac{s}{2}\beta\sin\psi + \ldots$$

and

$$\lambda = \int_0^{\psi} \frac{\lambda c\,(1+\alpha\cos\psi+\beta\sin\psi)}{q\psi\cos\psi}\cdot \qquad (62b)$$

$$x - \frac{s}{2} = -\int_0^{\psi} \frac{\lambda c\,(1+\alpha\cos\psi+\beta\sin\psi)}{q\psi\sin\psi}\cdot \qquad (62a)$$

The equation can also be written as components

$$s = \int_0^{\psi} \frac{\lambda c\,(1+\alpha\cos\psi+\beta\sin\psi)}{q\psi}\cdot \qquad (62)$$

or

$$\frac{qs}{q\psi} = \frac{\lambda c\,(1+\alpha\cos\psi+\beta\sin\psi)}{q\psi}\,,$$

It follows that

$$-\frac{EIc}{r_x^2} = 0\,, \quad \frac{EIc}{r_y^2} = \alpha, \text{ and } \frac{EI}{r_\varphi^2} = c\,.$$

The constants can be replaced by

$$\frac{qs}{q\psi} = \sqrt{-\frac{EIc}{r_x^2}\sin\psi + \frac{EIc}{r_y^2}\cos\psi + \frac{EI}{r_\varphi^2}}\,.$$

Equation (61) multiplied by $q\psi$ and integrated shows

$$\frac{s}{I}\left(\frac{qs}{q\psi}\right)_c = -\frac{EI}{r_x^2}\sin\psi + \frac{EI}{r_y^2}\cos\psi + \frac{EI}{r_\varphi^2}\,,$$

This value substituted into equations (62), (62a), and (62b), and integrated, gives:

$$s = \frac{1}{rc}\left[\varphi - \frac{a}{2}\sin\varphi + \frac{b}{2}\cos\varphi - \frac{b}{2}\right], \tag{63}$$

$$x - \rho_0 = -\frac{1}{rc}\left[1 - \cos\varphi - \frac{a}{8}(1-\cos 2\varphi) - \frac{b}{4}(\varphi - \tfrac{1}{2}\sin 2\varphi)\right], \tag{63a}$$

and

$$y = \frac{1}{rc}\left[\sin\varphi - \frac{a}{4}(\varphi - \tfrac{1}{2}\sin\varphi) - \frac{b}{8}(1-\cos 2\varphi)\right]. \tag{63b}$$

These equations determine the parameters a, b, and c as functions of the angle Θ.

These equations applied to the ferrule yields

$$\Theta\rho_0 = L = \frac{1}{rc}\left\{\Theta + \theta - \frac{a}{2}\sin(\Theta+\theta) + \frac{b}{2}\left[\cos(\Theta+\theta) - 1\right]\right\}, \tag{64}$$

$$\rho_0\cos(\Theta+\theta) - \rho_0 = -\frac{1}{rc}\left\{1 - \cos(\Theta+\theta) - \frac{a}{8}(1-\cos 2(\Theta+\theta)) - \frac{b}{4}\left[(\Theta+\theta) - \tfrac{1}{2}\sin 2(\Theta+\theta)\right]\right\}, \tag{64a}$$

and

$$\rho_0\sin(\Theta+\theta) = \frac{1}{rc}\left[\sin(\Theta+\theta) - \frac{a}{4}\left((\Theta+\theta) - \tfrac{1}{2}\sin(\Theta+\theta)\right) - \frac{b}{8}\left(1-\cos 2(\Theta+\theta)\right)\right]. \tag{64b}$$

The only unknown present are a, b, and c. Once these are calculated, the above equations give φ for every point along the spring.

This value substituted into equations (62), (62a), and (62b), and integrated, gives:

$$S = \frac{1C}{T}\left[\psi - \frac{2}{3}a\sin\psi + \frac{2}{5}b\cos\psi - \frac{2}{5}b\right], \tag{63}$$

$$\lambda - \delta^0 = -\frac{1C}{T}\left[1 - \cos\psi - \frac{8}{3}a(1-\cos 2\psi) - \frac{b}{4}(\psi - \tfrac{1}{2}\sin 2\psi)\right], \tag{63a}$$

and

$$\lambda = \frac{1C}{T}\left[\sin\psi - \frac{4}{3}a(\psi - \tfrac{1}{2}\sin\psi) - \frac{8}{5}b(1-\cos 2\psi)\right]. \tag{63b}$$

These equations determine the parameters a, b, and c as functions of the angle Θ.

These equations applied to the ferrule yields

$$\Theta\delta^0 = \mu = \frac{1C}{T}\left\{\Theta + \theta - \frac{2}{3}a\sin(\Theta+\theta) + \frac{2}{5}b\left[\cos(\Theta+\theta) - 1\right]\right\}, \tag{64}$$

$$\delta^0\cos(\Theta+\theta) - \delta^0 = -\frac{1C}{T}\left\{1 - \cos(\Theta+\theta) - \frac{8}{3}a(1-\cos 2(\Theta+\theta))\right.$$

$$\left. - \frac{b}{4}\left[(\Theta+\theta) - \frac{1}{2}\sin 2(\Theta+\theta)\right]\right\}, \tag{64a}$$

and

$$\delta^0\sin(\Theta+\theta) = \frac{1C}{T}\left[\sin(\Theta+\theta) - \frac{4}{3}a\left((\Theta+\theta) - \frac{1}{2}\sin(\Theta+\theta)\right)\right.$$

$$\left. - \frac{8}{5}b\left(1-\cos 2(\Theta+\theta)\right)\right]. \tag{64b}$$

The only unknown present are a, b, and c. Once these are calculated, the above equations give ψ for every point along the spring.

If the spring considered is one with involutes (a spiral), *(4)

$$(1+a\cos\varphi+b\sin\varphi)^{-1/2} = \sum_{n=0}^{\infty}(A_n\cos n\varphi + B\sin n\varphi).$$

The equations (63), (63a), and (63b) give

$$s = \frac{1}{rc}\left[A_0\varphi + \sum_{n=1}^{\infty}\frac{1}{n}(A_n\sin n\varphi - B_n\cos n\varphi + B)\right],$$

$$x-\rho_0 = -\frac{1}{rc}\left\{A_0(1-\cos\varphi) + \frac{A_1}{2}\sin^2\varphi + \frac{B_1}{2}\left(\varphi - \frac{\sin 2\varphi}{2}\right)\right.$$
$$+\frac{1}{2}\sum_{n=2}^{\infty}\left\{\frac{1}{n-1}\left[A_n\cos(n-1)\varphi + B_n\sin(n-1)\varphi - A_n\right]\right.$$
$$\left.\left.-\frac{1}{n+1}\left[A_n\cos(n+1)\varphi + B_n\sin(n+1)\varphi - A_n\right]\right\}\right],$$

and

$$y = \frac{1}{rc}\left[A_0\sin\varphi + \frac{A_1}{2}\left(\varphi+\sin\frac{\varphi}{2}\right) + \frac{B_1}{2}\sin^2\varphi\right.$$
$$+\frac{1}{2}\sum_{n=2}^{\infty}\left\{\frac{1}{n-1}\left[A_n\sin(n-1)\varphi + B_n\cos(n-1)\varphi - B_n\right]\right.$$
$$\left.\left.+\frac{1}{n+1}\left[A_n\sin(n+1)\varphi - B_n\cos(n+1)\varphi + B_n\right]\right\}\right].$$

To determine A_n and B_n, it would be necessary to expand by binomial expansion

$$(1+a\cos\varphi+b\sin\varphi)^{-1/2} = \sum_{n=0}^{\infty}(-\tfrac{1}{2})^n\frac{(2n-1)!}{n!}[a\cos\varphi+b\sin\varphi]^n.$$

If the spring considered is one with involutes (a spiral).

*(1)

$$(1+a\cos\psi+b\sin\psi)^{-n} = \sum_{m=0}^{\infty} (A_m \cos m\psi + B_m \sin m\psi).$$

The equations (63), (63a), and (63b) give

$$\delta = \frac{1}{EG}\left[A_0\psi + \sum_{m=1}^{\infty}\frac{1}{m}(A_m \sin m\psi - B_m \cos m\psi + B)\right].$$

$$\chi - \beta_0 = -\frac{1}{EG}\left\{A_0(1-\cos\psi) + \frac{A_1}{2}\sin 2\psi + \frac{B_1}{2}(\psi - \frac{\sin 2\psi}{2})\right.$$

$$\left. + \frac{1}{2}\sum_{m=2}^{\infty}\left\{\frac{1}{m-1}\left[A_m \cos(m-1)\psi + B_m \sin(m-1)\psi - A_m\right]\right.\right.$$

$$\left.\left. - \frac{1}{m+1}\left[A_m \cos(m+1)\psi + B_m \sin(m+1)\psi - A_m\right]\right\}\right).$$

and

$$\lambda = \frac{1}{EG}\left\{A_0 \sin\psi + \frac{A_1}{2}(\psi + \frac{\sin 2\psi}{2}) + \frac{B_1}{2}\sin^2\psi\right.$$

$$\left. + \frac{1}{2}\sum_{m=2}^{\infty}\left\{\frac{1}{m-1}\left[A_m \sin(m-1)\psi + B_m \cos(m-1)\psi - B_m\right]\right.\right.$$

$$\left.\left. + \frac{1}{m+1}\left[A_m \sin(m+1)\psi - B_m \cos(m+1)\psi + B_m\right]\right\}\right).$$

To determine A_m and B_m, it would be necessary to expand by binomial expansion

$$(1 + a\cos\psi + b\sin\psi)^{-n} = \sum_{m=0}^{\infty}(-1)^m \frac{(2m-1)!}{m!}[a\cos\psi + b\sin\psi]^m.$$

The sine and cosine changed into the form

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2},$$

and

$$\sin\varphi = \frac{e^{i\varphi} - e^{i\varphi}}{2i},$$

and substituted into $(a\cos\varphi + b\sin\varphi)^n$, yields the equation

$$\left[\left(\frac{a-ib}{2}\right)e^{i\varphi} + \frac{(a+ib)}{2}e^{-i\varphi}\right]^n = \frac{(a-bi)^n}{2}e^{in\varphi} + n\frac{(a-bi)^{n-1}}{2}\left(\frac{a+bi}{2}\right)e^{i(n-2)\varphi} + \frac{n(n-1)}{2!}\left(\frac{a-bi}{2}\right)^{n-2}\left(\frac{a+bi}{2}\right)^{i(n-4)\varphi} + \cdots .$$

This can be expressed as

.

$$\frac{1}{2^n}\sum_{P=0}^{P=n}\frac{n(n-1)\cdots(n-P+1)}{P!}(a-bi)^{n-P}(a-bi)^{P}e^{i(n-2P)\varphi}.$$

To evaluate a, b, and c is very laborious.

For the simple case, the solution to equations (64), (64a), and (64b), arrived at by a simple but somewhat laborious calculation, gives

The sine and cosine changed into the form

$$\cos\psi = \frac{e^{i\psi} + e^{-i\psi}}{2},$$

and

$$\sin\psi = \frac{e^{i\psi} - e^{-i\psi}}{2i},$$

and substituted into $(a\cos\psi + b\sin\psi)^n$ yields the equation

$$\left[\left(\frac{a}{2} - \frac{b}{2i}\right)e^{i\psi} + \left(\frac{a}{2} + \frac{b}{2i}\right)e^{-i\psi}\right]^n = \left(\frac{a}{2} - \frac{b}{2i}\right)^n e^{in\psi}$$

$$+ n\left(\frac{a}{2} - \frac{b}{2i}\right)^{n-1}\left(\frac{a}{2} + \frac{b}{2i}\right)e^{i(n-2)\psi}$$

$$+ \frac{n(n-1)}{2!}\left(\frac{a}{2} - \frac{b}{2i}\right)^{n-2}\left(\frac{a}{2} + \frac{b}{2i}\right)^2 e^{i(n-4)\psi} + \cdots$$

This can be expressed as

$$\frac{1}{2^n}\sum_{p=n}^{p=0} \frac{n(n-1)\cdots(n-p+1)}{p!}(a - bi)^{n-p}(a - bi)^p e^{i(n-2p)\psi}.$$

To evaluate a, b, and c is very laborious.

For the simple case, the solution to equations (61), (61a), and (61b), arrived at by a simple but somewhat laborious calculation, gives

$A = \rho_0(1-\cos\varphi) - \frac{\bar{L}}{A}(\varphi - \sin\varphi)$ when

$\textcircled{H} + \Theta$ is replaced by φ . Further results

$$Aa = \rho_0 \textcircled{H} \sin\varphi, \quad Ab = \rho\Theta(1-\cos\varphi), \tag{65}$$

and

$$A\overline{rc} = A\varphi - \rho_0\Theta(1-\cos\varphi) = \frac{A}{\bar{L}}(\varphi - b),$$

or

$$\overline{rc} = \frac{\varphi - b}{\bar{L}}.$$

If Θ is small compared with $\textcircled{H}$, then,

$$\textcircled{H}^2 a = 4\Theta\sin\varphi, \quad \textcircled{H}^2 b = 4\Theta(1-\cos\varphi), \tag{66}$$

$$\overline{rc}\,L = \varphi - b, \text{ and } \quad c\bar{L}\rho_0 = \frac{(\varphi - b)^2}{\textcircled{H}}.$$

The problem of Resal expresses primarily that the spring can be designed such that the reaction force changes the period by an amount independent of the aptitude of the balance wheel. In this case the reaction force does not have to be small.

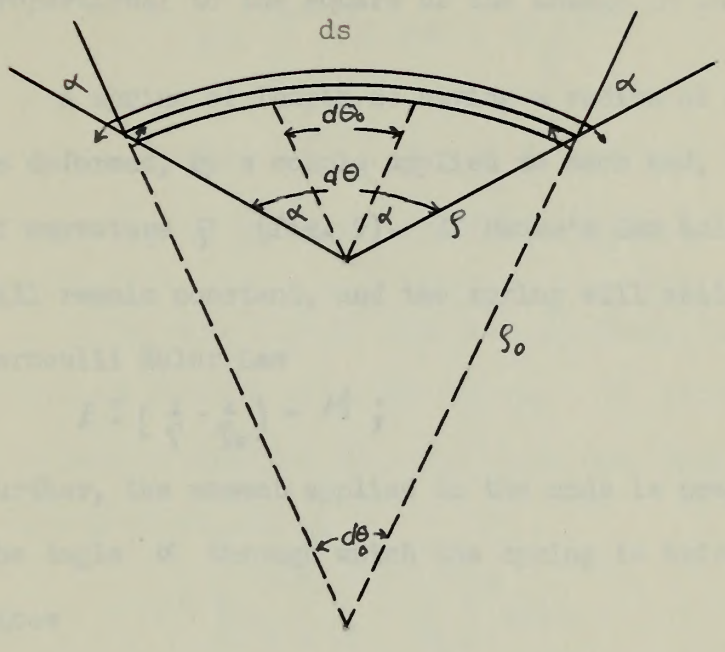
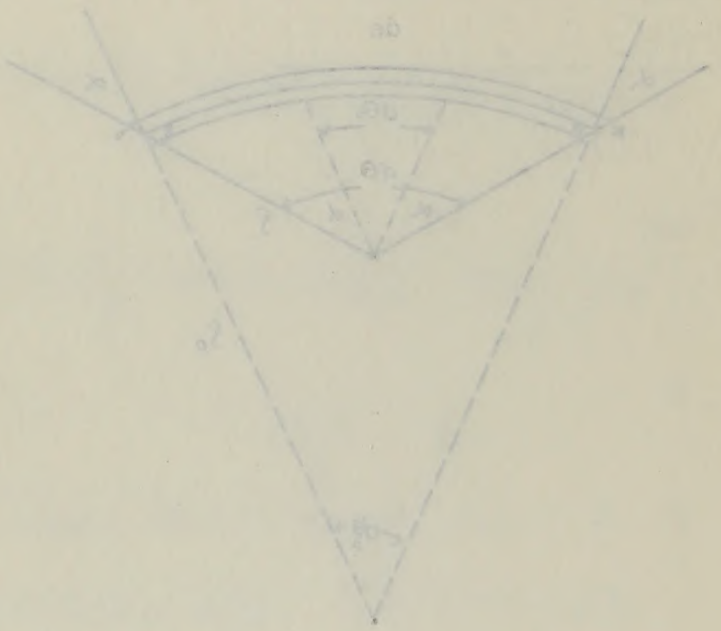


FIGURE 9. SECTION OF A DISTORTED SPRING



9. Elastic Energy

If a spring is deformed according to Hooke's Law, the energy stored in each element of the distorted spring is proportional to the square of the change in curvature.

A spring of length ds having a radius of curvature ρ_0 is deformed, by a couple applied to each end, to a radius of curvature ρ (fig. 9). If Hooke's Law holds, the center will remain constant, and the spring will satisfy the Bernoulli Euler Law

$$EI\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)=M;$$

further, the moment applied to the ends is proportional to the angle α through which the spring is twisted.

Since

$$ds=\rho\, d\theta,$$

and

$$ds=\rho_0\, d\theta_0,$$

$d\theta$ can be written as $d\theta = d\theta_0 + 2\alpha$.

This equation divided by ds gives

$$\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)\frac{ds}{2}=\alpha,$$

or

$$\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)A=\alpha, \qquad (67)$$

9. Elastic Energy

If a spring is deformed according to Hooke's law, the energy stored in each element of the distorted spring is proportional to the square of the change in curvature.

A spring of length ds having a radius of curvature ρ_0 is deformed, by a couple applied to each end, to a radius of curvature ρ (Fig. 9). If Hooke's law holds, the centre will remain constant, and the ends will satisfy the Bernoulli Euler law

$$EI\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) = M ;$$

further, the moment applied to the ends is proportional to the angle α through which the spring is bisected.

Thus

$$d\alpha = \frac{1}{2} d\theta ,$$

and

$$d\alpha = \frac{1}{2} d\theta_0 ,$$

$d\theta$ can be written as $d\theta = d\theta_0 + 2\alpha$.

This equation divided by ds gives

$$\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\frac{d\alpha}{ds} = \kappa ,$$

or

$$\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\frac{1}{2} = \kappa ,$$ (47)

because ds is constant.

Since the moment is proportional, the angle α, the work done upon an element ds, is

$$W = \int_0^{\alpha} \beta \alpha \, d\alpha = \beta \frac{\alpha^2}{2},$$

or

$$W = \frac{\beta A^2}{2} \left[\frac{1}{\rho} - \frac{1}{\rho_0} \right]^2, \tag{68}$$

where β is the proportionality constant.

The work done is also the potential energy stored in each element ds of the spring. The total energy stored is equal to the total work done upon the spring, and is expressed as

$$V = \frac{\xi}{2} \int_0^{\bar{L}} \left[\frac{1}{\rho} - \frac{1}{\rho_0} \right]^2 ds. \tag{69}$$

ξ depends on the material of the spring. and its cross section; $\bar{L}$ is the total length of the spring.

because ds is constant.

Since the moment is proportional, the angle α ,

the work done upon an element ds, is

$$M = \int_0^{\alpha} \beta \alpha \, d\alpha = \frac{\beta}{2} \alpha^2 ,$$

or

$$M = \frac{\beta}{2} A^2 \left[\frac{1}{\rho} - \frac{1}{\rho_0} \right]^2 , \qquad (38)$$

where β is the proportionality constant.

The work done is also the potential energy stored in each element ds of the spring. The total energy stored is equal to the total work done upon the spring, and is expressed as

$$V = \frac{\beta}{2} \int_0^L \left[\frac{1}{\rho} - \frac{1}{\rho_0} \right]^2 ds . \qquad (39)$$

β depends on the material of the spring, and its cross section; L is the total length of the spring.

10. Notation

Figure 10 represents a coordinate system*(2) having its origin on the axis of the balance wheel, and the inner end point of the spring is taken to be on the circumference of the balance wheel. The shape of the spring is characterized by giving at each point the angle ψ which the tangent makes with the x axis as a function of s, the arc length, measured from the inner end point.

From $ds^2 = dx^2 + dy^2$ the formulas for the shape of the spring become

$$dx = ds \cos\psi,$$

and

$$dy = ds \sin\psi, \tag{70}$$

These equations integrated become

$$x(s) = x(0) + \int_0^s ds \cos\psi(s), \tag{70a}$$

and

$$y(s) = y(0) + \int_0^s ds \sin\psi(s). \tag{70b}$$

These formulas are simplified by combining x and y into a complex number

$$x + iy.$$

10. Notation

Figure 10 represents a coordinate system*(2) having its origin on the axis of the balance wheel, and the inner end point of the spring is taken to be on the circumference of the balance wheel. The shape of the spring is characterized by giving at each point the angle ψ which the tangent makes with the x axis as a function of s, the arc length, measured from the inner end point.

From $ds_s^2 = dx_s^2 + dy_s^2$ the formulas for the shape of the spring become

$$dx = ds \cos\psi,$$

and

$$dy = ds \sin\psi, \tag{10}$$

These equations integrated become

$$x(s) = x(o) + \int_o^s ds \cos\psi(s), \tag{10a}$$

and

$$y(s) = y(o) + \int_o^s ds \sin\psi(s). \tag{10b}$$

These formulas are simplified by combining x and y into a complex number

$$x + iy.$$

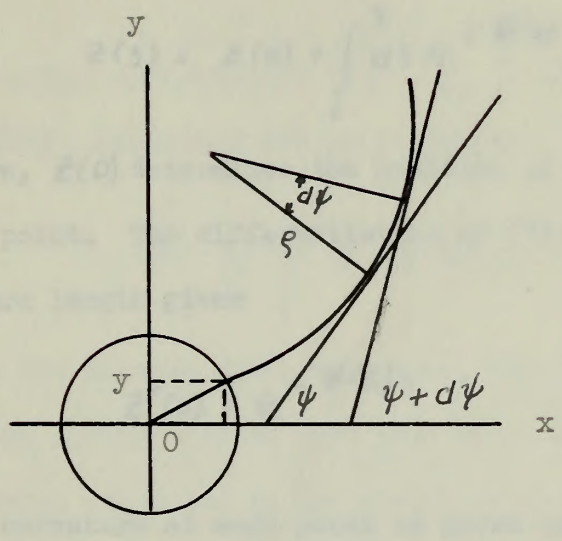
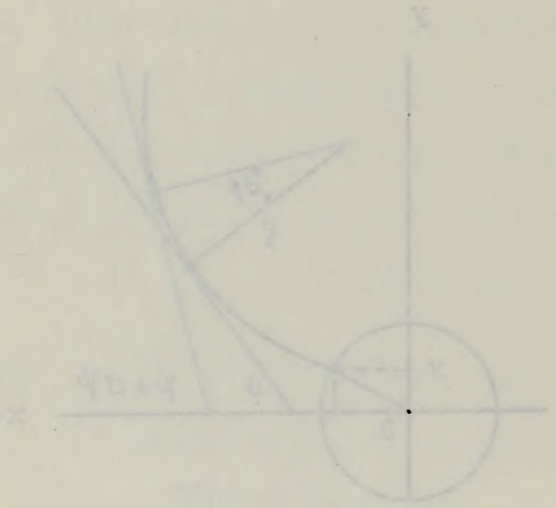


FIGURE 10. COORDINATES CHARACTERIZING THE FLAT SPRING



In this notation, the equation of the spring becomes

$$x + iy = x(0) + iy(0) + \int_0^S ds\,[\cos\psi(s) + i\sin\psi(s)],$$

or

$$z(s) = z(0) + \int_0^S ds\, e^{i\psi(s)}; \tag{71}$$

where, $z(0)$ determines the position of the inner end point. The differentiation of (71) with respect to arc length gives

$$z'(s) = e^{i\psi(s)}.$$

The curvature at each point is given by

$$\frac{1}{\rho} = \psi'(s).$$

When the balance wheel is rotated over an angle, the spring will distort and the new shape can be characterized by giving the change in $\psi(s)$. This change of angle for each element is denoted by $\phi(s)$.

Before variation $\psi = \psi_0(s)$,

and after variation $\psi = \psi_0(s) + \phi(s)$.

Differentiation gives

$$\frac{d\psi(s)}{ds} = \frac{d\psi_0(s)}{ds} + \frac{d\phi(s)}{ds},$$

and with $\frac{d\psi(s)}{ds}$ and $\frac{d\psi_c(s)}{ds}$ in terms of the radius of curvature gives

$$\left[\frac{1}{\rho} - \frac{1}{\rho_0}\right] = \phi'(s). \tag{72}$$

The change in curvature $\left[\frac{1}{\rho} - \frac{1}{\rho_0}\right]$ used in terms of $\phi(s)$, gives the energy equation

$$V = \frac{E}{2} \int_0^{\bar{L}} \phi'^2(s)\, ds. \tag{73}$$

When the balance wheel is turned, the distorted shape of the spring will be such that the energy V is a minimum, subjected to the proper conditions at the two end points. Then, $\phi(s)$ can be determined by

$$\delta V = \delta \left\{ \frac{E}{2} \int_0^{\bar{L}} \phi'^2(s)\, ds \right\} = 0.$$

The Euler Lagrange equation takes the form

$$\frac{d}{ds}\left(\frac{d\phi'^2(s)}{d\phi'}\right) - \frac{d\phi'^2(s)}{d\phi} = 0; \text{ or } \frac{d}{ds}(2\phi'(s)) = 0.$$

This expression integrated once, leads to

$$2\phi'(s) = 2A,$$

and integrated a second time, gives

$$\phi = As + B. \tag{74}$$

and with $\frac{d\psi(s)}{ds}$ and $\frac{d\phi(s)}{ds}$ in terms of the radius of curvature gives

$$\left[\frac{1}{\rho} - \frac{1}{\rho_o}\right] = \psi'(s) . \tag{12}$$

The change in curvature $\left[\frac{1}{\rho} - \frac{1}{\rho_o}\right]$ used in terms of $\psi(s)$, gives the energy equation

$$V = \frac{E}{2} \int_0^L \psi'^2(s)\, ds . \tag{13}$$

When the balance wheel is turned, the distorted shape of the spring will be such that the energy V is a minimum, subjected to the proper conditions at the two end points. Then, $\psi(s)$ can be determined by

$$\delta V = \delta \left\{ \frac{E}{2} \int_0^L \psi'^2(s)\, ds \right\} = 0 .$$

The Euler Lagrange equation takes the form

$$\frac{d}{ds}\left(\frac{d\psi'^2(s)}{d\psi'}\right) - \frac{d\psi^2(s)}{d\psi} = 0 \; ; \text{ or } \frac{d}{ds}\left(2\psi'(s)\right) = 0 .$$

This expression integrated once, leads to

$$2\psi'(s) = 2A ,$$

and integrated a second time, gives

$$\psi = As + B . \tag{14}$$

11. Free End Spring

When the balance wheel is turned over an angle α, the inner end point of the spring moves in a circle over the same arc*(2) (fig. 11); thus,

$$z_d(0) = z(0)\, e^{i\alpha}.$$

After distortion, $z(s)$ and $\phi(s)$ are denoted by

$$z_d(s) \quad \text{and} \quad \phi_d(s).$$

The definition of the free end spring requires that the tangent to the outer end point remains the same after distortion as before distortion.

The angle through which the tangent at the inner end point turns is chosen to be α; therefore,

$$\phi_d(l) = 0$$

and

$$\phi_d(0) = \alpha. \tag{75}$$

From these boundary conditions (75), the equation $\phi = As + B$ for the free spring becomes

$$\phi(s) = -\frac{\alpha s}{l} + \alpha, \tag{76}$$

and

$$\phi'(s) = -\frac{\alpha}{l}.$$

11. Free End Spring

When the balance wheel is turned over an angle α, the inner end point of the spring moves in a circle over the same arc[2] (fig. 11): thus,

$$\xi^d(o) = \xi(o) \, \varepsilon_{,\alpha}.$$

After distortion, $\xi(s)$ and $\phi(s)$ are denoted by

$$\xi^d(s) \quad \text{and} \quad \phi^d(s).$$

The definition of the free end spring requires that the tangent to the outer end point remains the same after distortion as before distortion.

The angle through which the tangent at the inner end point turns is chosen to be α ; therefore,

$$\phi^d(\bar{\mu}) = o$$

and (75)

$$\phi^d(o) = \alpha .$$

From these boundary conditions (75), the equation $\Phi = Ae + B$ for the free spring becomes

$$\phi(s) = -\frac{\mu}{\alpha} s + \alpha \; , \tag{76}$$

and

$$\phi'(s) = -\frac{\mu}{\alpha} .$$

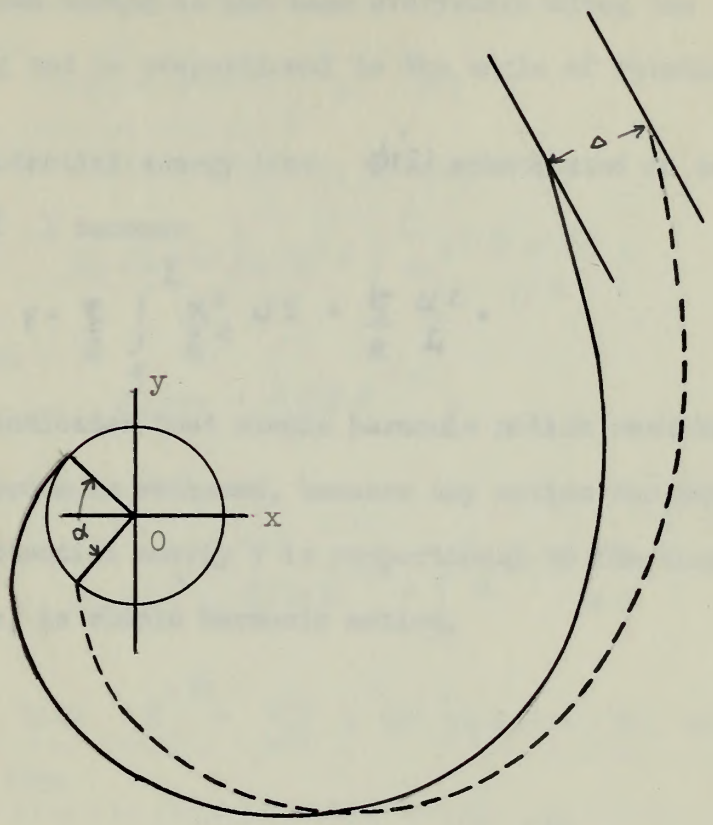
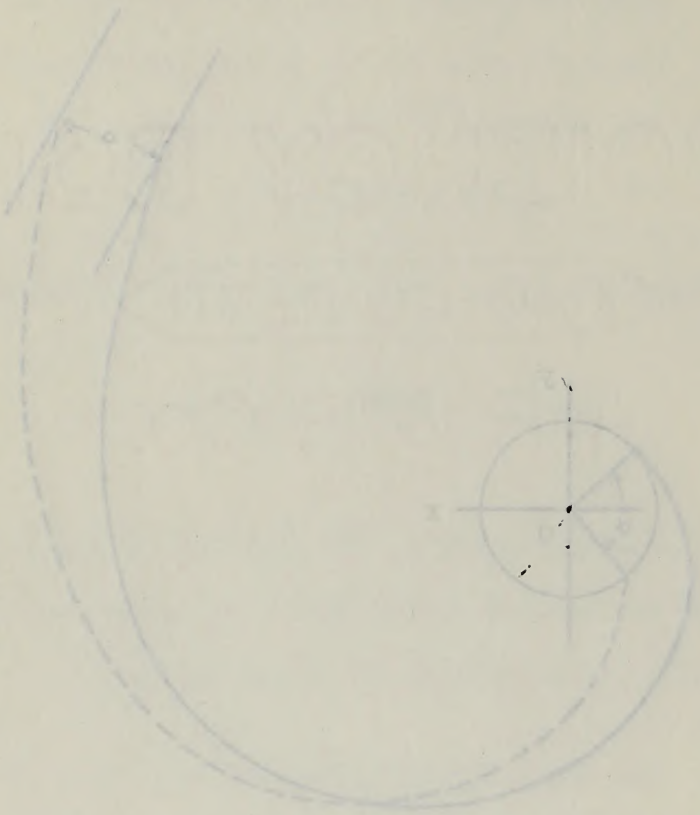


FIGURE 11. DISTORTED FLAT SPRING



This expresses that the change in curvature for the free end spring is the same everywhere along the spring and is proportional to the angle of rotation α .

The potential energy (for $\phi'(s)$ substituted in terms of α) becomes

$$V = \frac{\varepsilon}{2} \int_0^{\bar{L}} \frac{\alpha^2}{\bar{L}^2} \, ds = \frac{\varepsilon}{a} \frac{\alpha^2}{\bar{L}} . \qquad (77)$$

This indicates that simple harmonic motion results if the torque is released, because any motion for which the potential energy V is proportional to the displacement square, is simple harmonic motion.

This expresses that the change in curvature for the free end spring is the same everywhere along the spring and is proportional to the angle of rotation α.

The potential energy (for $\phi(s)$ substituted in terms of α) becomes

$$V = \frac{E}{2} \int_0^L \frac{\alpha_s}{L}^2 \, ds = \frac{E}{2} \frac{a}{L} \alpha_s^2 . \qquad (77)$$

This indicates that simple harmonic motion results if the torque is released, because any motion for which the potential energy V is proportional to the displacement square, is simple harmonic motion.

12. End Point Displacement

At $S = \bar{L}$ the formula*(2)

$$z(s) = z(0) + \int_0^s e^{i\psi} ds$$

for the distorted spring, becomes

$$z_d(\bar{L}) = z_d(0) + \int_0^{\bar{L}} e^{i(\psi+\phi)} \cdot ds \qquad (78)$$

Since

$$z_d(0) = z(0) e^{i\alpha},$$

it can be written as

$$z_d(\bar{L}) = z(0) e^{i\alpha} + \int_0^{\bar{L}} e^{i(\psi+\phi)} ds . \qquad (79)$$

With $e^{i\psi} = \frac{dz}{ds}$, the equation (79) takes the form

$$z_d(\bar{L}) = z(0) e^{i\alpha} + \int_0^{\bar{L}} e^{i\phi} dz . \qquad (80)$$

It follows that the expression $\int_0^{\bar{L}} e^{i\phi} dz$,

integrated by parts, results in

$$\int_0^{\bar{L}} e^{i\phi} dz = z e^{i\phi} \Big|_0^{\bar{L}} - i \int_0^{\bar{L}} z e^{i\phi} d\phi .$$

This expression rewritten in the form

$$z e^{i\phi} \Big|_0^{\bar{L}} - i \int_0^{\bar{L}} z e^{i\phi} \frac{d\phi}{ds} ds ,$$

12. End Point Displacement

At $s = L$ the formula*(5)

$$\Sigma(s) = \Sigma(0) + \int_0^s \varepsilon\, e^{i\psi}\, dz$$

for the distorted spring becomes

$$\Sigma^q(L) = \Sigma^q(0) + \int_0^L \varepsilon\, e^{i(\psi+\phi)}\, \frac{dz}{ds}\,. \tag{78}$$

Since

$$\Sigma^q(0) = \Sigma(0)\, e^{i\alpha}\,,$$

it can be written as

$$\Sigma^q(L) = \Sigma(0)\, e^{i\alpha} + \int_0^L \varepsilon\, e^{i(\psi+\phi)}\, \frac{dz}{ds}\,. \tag{79}$$

With $\varepsilon\, e^{i\psi} = \frac{dz}{ds}$, the equation (79) takes the form

$$\Sigma^q(L) = \Sigma(0)\, e^{i\alpha} + \int_0^L \varepsilon\, e^{i\phi}\, dz\,. \tag{80}$$

It follows that the expression $\int_0^L \varepsilon\, e^{i\phi}\, dz$,

integrated by parts, results in

$$\int_0^L \varepsilon\, e^{i\phi}\, dz = z\, \varepsilon\, e^{i\phi}\Big|_0^L - i \int_0^L z\, \varepsilon\, e^{i\phi}\, d\phi\,.$$

This expression rewritten in the form

$$z\, \varepsilon\, e^{i\phi}\Big|_0^L - i \int_0^L z\, \varepsilon\, e^{i\phi}\, \frac{d\phi}{ds}\, dz\,,$$

and with

$$\frac{d\phi}{ds} = -\frac{\alpha}{\bar{L}}, \quad \text{and} \quad \phi = -\frac{\alpha}{\bar{L}} s + \alpha ,$$

the solution becomes

$$\int_0^{\bar{L}} e^{i\phi} dz = z e^{i\left(-\frac{\alpha}{\bar{L}} s + \alpha\right)} \Big|_0^{\bar{L}} - i \int_0^{\bar{L}} z\, e^{i\left(-\frac{\alpha}{\bar{L}} s + \alpha\right)} \left(-\frac{\alpha}{\bar{L}}\right) ds ,$$

or

$$\int_0^{\bar{L}} e^{i\phi} ds = \left[z(\bar{L}) - z(0) e^{i\alpha}\right] + \frac{i\alpha}{\bar{L}} e^{i\alpha} \int_0^{\bar{L}} z\, e^{-i\frac{\alpha s}{\bar{L}}} ds .$$

From the solution of $\int_0^{\bar{L}} e^{i\phi} dz$, replaced into the original equation (80), arises the expression

$$z_d(\bar{L}) = z(0) e^{i\alpha} + z(\bar{L}) - z(0) e^{i\alpha} + \frac{i\alpha}{\bar{L}} e^{i\alpha} \int_0^{\bar{L}} z\, e^{-i\frac{\alpha s}{\bar{L}}} ds .$$

From the displacement of the outer end point,

$\Delta = z_d(\bar{L}) - z(\bar{L})$, arises the general formula

$$\Delta = \frac{i\alpha}{\bar{L}} e^{i\alpha} \int_0^{\bar{L}} z\, e^{-i\frac{\alpha s}{\bar{L}}} ds . \qquad (81)$$

The relation (81) is valid for the free end and the fastened spring, because the relations

The relation (81) is valid for the free end, and the fastened spring, because the relations

$$\Delta = \frac{i\alpha}{\bar{\mu}} \int_0^{\bar{\mu}} z\, e^{i\alpha \frac{\bar{\mu}}{\bar{\mu}} s}\, ds \,. \qquad (81)$$

arises the general formula

$$\Delta = z_q(\bar{\mu}) - z(\bar{\mu})$$

From the displacement of the outer end point,

$$z_q(\bar{\mu}) = z(o)\, e^{i\alpha} + z(\bar{\mu}) - z(o)\, e^{i\alpha} + i\alpha\, e^{i\alpha} \int_0^{\bar{\mu}} z\, e^{-i\alpha \frac{\bar{\mu}}{\bar{\mu}} s}\, ds \,.$$

From the solution of $\int_0^{\bar{\mu}} e^{i\phi}\, d\xi$, replaced into the original equation (60), arises the expression

$$\int_0^{\bar{\mu}} e^{i\phi}\, dz = \left[z(\bar{\mu}) - z(o)\, e^{i\alpha} \right] + i\alpha\, e^{i\alpha} \int_0^{\bar{\mu}} z\, e^{-i\alpha \frac{\bar{\mu}}{\bar{\mu}} s}\, ds \,.$$

or

$$\int_0^{\bar{\mu}} e^{i\phi}\, dz = z\, e^{i\left(-\frac{\bar{\mu}}{\alpha} s + \alpha\right)} - i \int_0^{\bar{\mu}} z\, e^{i\left(-\frac{\bar{\mu}}{\alpha} s + \alpha\right)}\, ds \,,$$

the solution becomes

$$\frac{d\phi}{ds} = -\frac{\bar{\mu}}{\alpha} \,, \quad \text{and} \quad \phi = -\frac{\bar{\mu}}{\alpha} s + \alpha \,,$$

and with

$$\phi_d(\bar{\lambda}) = 0$$

and

$$\phi_d(0) = \alpha$$

are true even for the fastened spring.

The problem consists in designing the undistorted spring $\bar{z}(s)$ so that the end point displacement, and hence, the integral becomes as small as possible.

$$\phi_a(\bar{u}) = 0$$

and

$$\phi_a(0) = \alpha$$

are true even for the fastened spring.

The problem consists in designing the unfastened spring $\Sigma(s)$ so that the end point displacement, and hence, the integral becomes as small as possible.

13. Circular Shaped "Helix"

The equation of a "Helical"*(2) spring of radius R, consisting of n complete windings and a partial winding of arc θ, is $z(s) = Re^{i\theta}$,

or $$z(s) = Re^{iS/R},$$

and the total length of the spring is $L = (2\pi n + \theta)R$.

The end point displacement becomes

$$\Delta = i\frac{\alpha}{L} e^{i\alpha} \int_0^L R e^{iS/R} e^{i\alpha S/L} dS,$$

$$\Delta = i\frac{\alpha}{L} e^{i\alpha} R \int_0^L e^{i\left(\frac{1}{R} - \frac{\alpha}{L}\right)S} dS,$$

$$\Delta = i\frac{\alpha}{L} e^{i\alpha} R \left[\frac{e^{i\left(\frac{1}{R} - \frac{\alpha}{L}\right)L} - 1}{i\left(\frac{1}{R} - \frac{\alpha}{L}\right)}\right],$$

or

$$\Delta = \frac{\alpha R^2 e^{i\alpha}}{(L - R\alpha)} \left[e^{i(L/R - \alpha)} - 1\right]. \qquad (82)$$

The value for L substituted into the expression (82), gives

$$\Delta = \frac{\alpha R}{[(2\pi n + \theta) - \alpha]} \left\{e^{i(2\pi n + \theta)} - e^{i\alpha}\right\}. \qquad (83)$$

This takes the form

$$\Delta = \frac{\alpha R}{[(2\pi n + \theta) - \alpha]} \left\{e^{i\theta} - e^{i\alpha}\right\} \simeq \frac{L\alpha}{4\pi^2 n} \left[e^{i\theta} - e^{i\alpha}\right].$$

13. Circular Shaped "Helix"

The equation of a "helical"*(2) spring of radius R, consisting of n complete windings and a partial winding of arc θ, is $\bar{z}(s) = R\,e^{i\frac{s}{R}}$,

or $z(s) = R\,e^{i\frac{s}{R}}$,

and the total length of the spring is $L = (2\pi n + \theta)R$.

The end point displacement becomes

$$\Delta = \frac{P}{\alpha}\left\{e^{i\alpha} \int_0^L R\, e^{i\frac{s}{R}} e^{i\alpha\frac{s}{R}}\, ds\right\},$$

$$\Delta = \frac{P}{\alpha} e^{i\alpha} \int_0^L e^{i\left(\frac{1}{R} - \frac{\alpha}{L}\right)s},$$

$$\Delta = \frac{P}{\alpha} e^{i\alpha} R \left[\frac{e^{i\left(\frac{L}{R} - \alpha\right)L} - 1}{i\left(\frac{L}{R} - \frac{\alpha}{L}\right)}\right],$$

or

$$\Delta = \frac{\alpha R^2 e^{i\alpha}}{(L - R\alpha)}\left[e^{i\left(\frac{L}{R} - \alpha\right)} - 1\right]. \qquad (82)$$

The value for L substituted into the expression (82), gives

$$\Delta = \frac{\alpha R}{\left[(2\pi n + \theta) - \alpha\right]}\left\{e^{i(2\pi n + \theta)} - e^{i\alpha}\right\}. \qquad (83)$$

This takes the form

$$\Delta = \frac{\alpha R}{\left[(2\pi n + \theta) - \alpha\right]}\left\{e^{i\theta} - e^{i\alpha}\right\} = \frac{4\pi^2 n}{\alpha}\left[e^{i\theta} - e^{i\alpha}\right].$$

For a free spring of one complete winding,

$$\Delta = \frac{\alpha R}{2\pi - \alpha}\left\{1 - e^{i\alpha}\right\},$$

or

$$\Delta = \frac{\alpha R}{2\pi - \alpha}\left\{1 - (\cos\alpha + i\sin\alpha)\right\};$$

hence,

$$\Delta x = \frac{\alpha R}{2\pi - \alpha}(1 - \cos\alpha)$$

and

$$\Delta y = \frac{\alpha R}{2\pi - \alpha}\sin\alpha.$$

14. Ellipse

When the fundamental equation for the displacement of the end point (81) is applied to springs characterized by various shapes, the integral becomes somewhat difficult to integrate. To illustrate this, an elliptical helix is considered.

In polar coordinates, the equation of an ellipse with the pole at the center of the coordinate system is represented by

$$r^2 = \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}, \tag{84}$$

where a and b are the semi axes. Any point on the ellipse is defined by $x = a \sin\theta$,

and

$$y = \frac{b}{a}\sqrt{a^2 - x^2} = b \cos\theta.$$

The arc length of the ellipse is represented by the expression

$$s = \int_0^\theta \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}\, d\theta.$$

With the $\cos^2\theta$ replaced by $1 - \sin^2\theta$, the integral becomes

$$s = a\int_0^\theta \sqrt{1 - k^2 \sin^2\theta}\, d\theta; \tag{85}$$

where, $k = \frac{\sqrt{a^2 - b^2}}{a}$ is the eccentricity of the

11. Ellipse

When the fundamental equation for the displacement of the end point (21) is applied to springs characterized by various shapes, the integral becomes somewhat difficult to integrate. To illustrate this, an elliptical helix is considered.

In polar coordinates, the equation of an ellipse with the pole at the center of the coordinate system is represented by

$$\rho_s = \frac{a_s b_s}{\sqrt{a_s^2 \sin^2\theta + b_s^2 \cos^2\theta}} \quad (84)$$

where a and b are the semi axes. Any point on the ellipse is defined by $x = a \sin\theta$,

and

$$Y = \frac{b}{a}\sqrt{a_s^2 - x_s^2} = b\cos\theta .$$

The arc length of the ellipse is represented by the expression

$$s = \int_0^\theta \sqrt{a_s^2\cos^2\theta + b_s^2\sin^2\theta}\, d\theta .$$

With the $\cos^2\theta$ replaced by $1 - \sin^2\theta$, the integral becomes

$$S = a \int_0^\theta \sqrt{1 - K_s^2\sin^2\theta}\, d\theta ; \quad (85)$$

where, $K = \frac{\sqrt{a_s^2 - b_s^2}}{a}$ is the eccentricity of the

ellipse and is less than unity.

In terms of θ the complex number $z = re^{i\theta}$ becomes

$$z = \frac{ab\,e^{i\theta}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} .$$

The end point displacement is now given by

$$\Delta = \frac{i\alpha}{N} e^{i\alpha} ab \int_0^{\theta_N} \frac{e^{i\theta}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\, e^{-\frac{i\alpha a}{N}\int_0^{\theta}\sqrt{1-k\sin^2\theta}\,d\theta}\, d\theta,$$

or

$$\Delta = \frac{i\alpha}{N} e^{i\alpha} a^2 b \int_0^{\theta_N} \frac{\sqrt{1-k^2\sin^2\theta}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\, e^{i\theta} e^{-\frac{i\alpha a}{N}\int_0^{\theta}\sqrt{1-k^2\sin^2\theta}\,d\theta}\, d\theta . \qquad (86)$$

This integral can be evaluated with the use of elliptic integral tables.

15. Logarithmic Spiral

In polar coordinates the equation of the logarithmic spiral is

$$r = e^{a\theta}.$$

It follows that in the complex number notation

$$z = e^{a\theta} e^{i\theta}.$$

The angle starts from zero and goes up to θ. The arc length is given by

$$S = \int_0^{\theta} e^{a\theta} \sqrt{1+a^2}\, d\theta = \frac{\sqrt{1+a^2}}{a}\left[e^{a\theta} - 1 \right]. \qquad (87)$$

For the end point, displacement is obtained, using θ as a variable,

$$\Delta = \frac{i\alpha}{L} e^{i\alpha} \int_0^{\theta_L} e^{2a\theta} e^{i\theta} e^{-i\frac{\alpha}{L}\frac{\sqrt{1+a^2}}{a}\left[e^{a\theta}-1\right]} \frac{\sqrt{1+a^2}}{} d\theta,$$

or

$$\Delta = \frac{i\alpha}{L} e^{i\alpha \frac{\left[1+\sqrt{\frac{1+a^2}{a}}\right]}{\sqrt{1+a^2}}} \int_0^{\theta_L} e^{(2a+i)\theta} e^{-i\left\{\frac{\alpha}{L}\sqrt{\frac{1+a^2}{a}} e^{a\theta}\right\}}. \qquad (88)$$

If a is very small, $e^{a\theta}$ can be written as $1 + a\theta$ if the second order terms are neglected.

Now, equation (88) can be written as

$$\Delta = i\frac{\alpha}{L} e^{i\alpha} \sqrt{1+a^2} \int_0^{\theta_L} e^{(2a+i)\theta} e^{-i\frac{\alpha}{L}\frac{\sqrt{1+a^2}}{a}(1+a\theta)}, \qquad (89)$$

where this integral has to be evaluated numerically.

15. Logarithmic Spiral

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$$s = \int_0^\theta e^{a\theta} \sqrt{1 + a^2}\, d\theta = \frac{\sqrt{1+a^2}}{a} \left[e^{a\theta} - 1 \right]. \quad (87)$$

For the end point, displacement is obtained, using θ as a variable,

$$\Delta = \frac{P}{4} e^{i\alpha} \int_0^\theta e^{2a\theta} e^{i\theta} e^{-i\alpha} \frac{(1+a^2)}{a} \left[e^{a\theta} - 1 \right] d\theta$$

or

$$\Delta = \frac{P}{4} e^{i\alpha} \left[1 + \frac{(1+a^2)}{a} \right] \int_0^\theta e^{(2a+1)\theta} e^{-i\left(\frac{P}{4} \frac{\sqrt{1+a^2}}{a} e^{i\alpha}\right)} \quad (88)$$

If a is very small, $e^{a\theta}$ can be written as $1 + a\theta$ if the second order terms are neglected.

Now, equation (88) can be written as

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where this integral has to be evaluated numerically.

16. Terminals

The spring*(2) of length $\bar{L}$ is divided into three parts ℓ_1, $\bar{L}-\ell_1-\ell_2$, and ℓ_2. The terminals ℓ_1 and ℓ_2 are short compared to the total length $\bar{L}$. The end point displacement is considered separately for each part, and the problem is to design the spring so that the sum of the contributions from the two end parts cancell the contribution from the center part, hence, the displacement $\Delta = 0$.

The contribution from ℓ_1 is

$$\Delta_1 = i\frac{\alpha}{\bar{L}} e^{i\alpha} \int_0^{\ell_1} z(s)\, e^{-i\alpha s/\bar{L}}\, ds ,$$

or

$$\Delta_1 = i\frac{\alpha}{\bar{L}} e^{i\alpha} \int_0^{\ell_1} z(\sigma)\, e^{-i\alpha(-\sigma+\ell_1)/\bar{L}}\, d\sigma , \qquad (90)$$

where $\sigma = \ell_1 - s$ is the arc length measured along this terminal in the opposite direction.

The contribution from ℓ_2 becomes

$$\Delta_2 = i\frac{\alpha}{\bar{L}} e^{i\alpha} \int_{\bar{L}-\ell_2}^{\bar{L}} z(s)\, e^{-i\alpha s/\bar{L}}\, ds ,$$

or

$$\Delta_2 = i\frac{\alpha}{\bar{L}} e^{i\alpha} \int_{\bar{L}-\ell_2}^{\bar{L}} z(\sigma)\, e^{-i\alpha(\sigma+\bar{L}-\ell_2)/\bar{L}}\, d\sigma ,$$

16. Terminals

The spring*(2) of length $\bar{L}$ is divided into three parts ℓ_1, $\bar{L} - \ell_1 - \ell_2$, and ℓ_2. The terminals ℓ_1 and ℓ_2 are short compared to the total length $\bar{L}$. The end point displacement is considered separately for each part, and the problem is to design the spring so that the sum of the contributions from the two end parts cancell the contribution from the center part, hence, the displacement $\Delta = 0$.

The contribution from ℓ_1 is

$$\Delta_1 = \epsilon \frac{\bar{\mu}}{\bar{L}} e^{i\omega t} \int_0^{\ell_1} \xi(s)\, e^{-\epsilon \frac{\bar{\mu}}{\bar{L}} s}\, ds ,$$

or

$$\Delta_1 = \epsilon \frac{\bar{\mu}}{\bar{L}} e^{i\omega t} \int_0^{\ell_1} \xi(Q)\, e^{-\epsilon \frac{\bar{\mu}}{\bar{L}} (-Q + \ell_1)}\, dQ , \qquad (20)$$

where $Q = \ell_1 - s$ is the arc length measured along this terminal in the opposite direction.

The contribution from ℓ_2 becomes

$$\Delta_2 = \epsilon \frac{\bar{\mu}}{\bar{L}} e^{i\omega t} \int_{\bar{L}-\ell_2}^{\bar{L}} \xi(s)\, e^{-\epsilon \frac{\bar{\mu}}{\bar{L}} s}\, ds ,$$

or

$$\Delta_2 = \epsilon \frac{\bar{\mu}}{\bar{L}} e^{i\omega t} \int_{\bar{L}-\ell_2}^{\bar{L}} \xi(Q)\, e^{-\epsilon \frac{\bar{\mu}}{\bar{L}} (Q + \bar{L} - \ell_2)}\, dQ ,$$

and finally results

$$\Delta_2 = \frac{i\alpha}{\bar{L}} e^{i\alpha \frac{\ell_2}{\bar{L}}} \int_0^{\ell_2} z(\sigma)\, e^{-i\alpha\frac{\sigma}{\bar{L}}}\, d\sigma \,. \qquad (91)$$

The latter integral is obtained if the arc length is measured from the beginning of the terminal, $\sigma = s - \bar{L} + \ell_2$.

If the central portion is taken to be a "helical" spring, its equation can be taken as

$$z(s) = R\, e^{i\left(s - \frac{\ell_1}{R}\right)},$$

The arc length of the "helical" part goes from

$$\ell_1 \quad \text{to} \quad \bar{L} - \ell_2 \,.$$

The helix is considered to be made up of n complete windings plus an arc θ ; hence,

$$\bar{L} - \ell_1 - \ell_2 = (2\pi n + \theta) R \,.$$

The displacement for the center portion is

$$\Delta_c = \frac{i\alpha}{\bar{L}} e^{i\alpha} \int_{\ell_1}^{\bar{L}-\ell_2} R\, e^{-\left(\frac{s-\ell_1}{R}\right)} e^{-i\alpha\frac{s}{\bar{L}}}\, ds \,,$$

$$\Delta_c = \frac{i\alpha}{\bar{L}} e^{i\alpha} R\, e^{-i\frac{\ell_1}{R}} \int_{\ell_1}^{\bar{L}-\ell_2} e^{i\left[1/R - \frac{\alpha}{\bar{L}}\right]s}\, ds \,,$$

$$\Delta_c = \frac{\frac{\alpha}{\bar{L}} R e^{i\alpha} e^{-i\frac{\ell_1}{R}} e^{i\left[1/R - \alpha/L\right]s}}{\left[1/R - \alpha/L\right]} \Bigg|_{\ell_1}^{\bar{L}-\ell_2}, \qquad (92)$$

and finally results

$$\Delta_3 = \left(i\frac{\mu}{E}\right) e^{i\alpha\frac{k_3}{R}} \int_0^{k_3} e^{(\alpha-i\delta)\xi} \, d\xi \,. \qquad (31)$$

The latter integral is obtained if the arc length is measured from the beginning of the terminal, $\xi = S - L + k_3$.

If the central portion is taken to be a "helical" spring, its equation can be taken as

$$\xi(S) = R\,\varepsilon_1\left(\frac{S - k_1}{R}\right),$$

The arc length of the "helical" part goes from

$$k_1 \quad \text{to} \quad L - k_3 \,.$$

The helix is considered to be made up of n complete windings plus an arc Θ ; hence,

$$L - k_1 - k_3 = (2\pi n + \Theta)R \,.$$

The displacement for the center portion is

$$\Delta_2 = \left(i\frac{\mu}{E}\right) e^{i\alpha\frac{k_1}{R}} \int_{k_1}^{L-k_3} R\,\varepsilon_1\left(\frac{S-k_1}{R}\right) e^{-i\delta\frac{\xi}{R}} \, dS \,,$$

$$\Delta_2 = \left(i\frac{\mu}{E}\right) e^{i\alpha\frac{k_1}{R}} R\,\varepsilon_1\left[\left(\frac{L-k_1-k_3}{R}\right)S\right] \int_{k_1}^{L-k_3} e^{-i\delta\frac{\xi}{R}} \, dS \,,$$

$$\Delta_2 = \left(i\frac{\mu}{\alpha}\right) R\,\varepsilon_1\left[\left(\frac{L-k_1-k_3}{R}\right)S\right] \left[e^{i\alpha\frac{k_1}{R}} - e^{-i\delta\frac{k_3}{R}}\right] \,. \qquad (32)$$

$$\Delta_c = \frac{\alpha R^2 e^{i\alpha}}{[L-\alpha R]} \left\{ e^{-i\frac{l_1}{R}} e^{i\left[\frac{L-l_2}{R} - \frac{\alpha}{L}(L-l_2)\right]} - e^{-i\frac{l_1}{R}} e^{i\left[\frac{l_1}{R} - \frac{\alpha l_1}{L}\right]} \right\},$$

$$\Delta_c = \frac{\alpha R^2 e^{i\alpha}}{[L-\alpha R]} \left\{ e^{i(2n\pi+\theta)} e^{i\left[-\alpha + \frac{\alpha l_2}{L}\right]} - e^{-i\frac{\alpha l_1}{L}} \right\},$$

$$\Delta_c = \frac{\alpha R^2 e^{i\alpha}}{[L-\alpha R]} \left\{ e^{i\theta} e^{-i\alpha} e^{i\alpha\frac{l_2}{L}} - e^{-i\alpha\frac{l_1}{L}} \right\},$$

or finally

$$\Delta_c = \frac{\alpha R^2}{[L-\alpha R]} \left[e^{i\theta} e^{i\alpha\frac{l_2}{L}} - e^{i\alpha} e^{-i\frac{\alpha}{L} l_1} \right]. \qquad (93)$$

In order that $\Delta_1 + \Delta_2 + \Delta_c \simeq 0$, the solution for Δ_c must be compared with the expressions Δ_1 and Δ_2.

The equations (90)

$$\Delta_1 = i\frac{\alpha}{L} e^{i\alpha} e^{-i\frac{\alpha l_1}{L}} \int_0^{l_1} z(\sigma) e^{i\sigma\frac{\alpha}{L}} d\sigma,$$

and (91)

$$\Delta_2 = i\frac{\alpha}{L} e^{i\alpha\frac{l_2}{L}} \int_0^{l_2} z(\sigma) e^{-i\frac{\alpha\sigma}{L}} d\sigma,$$

$$\Delta^s = \frac{\pi}{\alpha} e^{i\alpha \frac{\pi}{t_s}} \begin{cases} \varepsilon(\alpha) e^{-i\alpha \frac{\pi}{t_s} q_\alpha} \\ 0 \end{cases},$$

and (91)

$$\Delta' = \frac{\pi}{\alpha} e^{i\alpha} e^{-i\alpha \frac{\pi}{t'}} \begin{cases} \varepsilon(\alpha) e^{i\alpha \frac{\pi}{t'} q_\alpha} \\ 0 \end{cases},$$

The equations (90)

Δ^s.

for Δ^c must be compared with the expressions Δ' and

In order that $\Delta' + \Delta^s + \Delta^c = 0$, the solution

$$\Delta^c = \frac{[\pi - \alpha k]}{\alpha k_s} \left[e^{i\theta} e^{i\alpha \frac{\pi}{t_s}} - e^{i\alpha} e^{-i\alpha \frac{\pi}{t'}} \right] . \quad (93)$$

or finally

$$\Delta^c = \frac{[\pi - \alpha k]}{\alpha k_s e^{i\alpha}} \begin{cases} e^{i\theta} e^{-i\alpha} e^{i\alpha \frac{\pi}{t_s}} - e^{-i\alpha \frac{\pi}{t'}} \\ \end{cases},$$

$$\Delta^c = \frac{[\pi - \alpha k]}{\alpha k_s e^{i\alpha}} \begin{cases} e^{i(3u\pi + \theta)} e^{i[-\alpha + \alpha \frac{t}{t_s}]} - e^{-i\alpha \frac{\pi}{t'}} \\ \end{cases},$$

$$\Delta^c = \frac{[\pi - \alpha k]}{\alpha k_s e^{i\alpha}} \begin{cases} e^{-i\frac{k}{t'}} e^{i[\frac{k}{t_s} - \frac{\pi}{\alpha}(\pi - t_s)]} \\ - e^{-i\frac{k}{t'}} e^{i[\frac{k}{t'} - \alpha \frac{\pi}{t'}]} \end{cases},$$

Compared with (93)

$$A_c = \frac{\alpha R^2}{[\lambda - \alpha R]} \left[e^{i\theta} e^{i\alpha \frac{l_2}{L}} - e^{i\alpha} e^{-i\alpha \frac{l_1}{L}} \right],$$

gives

$$\frac{e^{i\theta} R^2 \alpha}{[\lambda - R\alpha]} \simeq -i \frac{\alpha}{L} \int_0^{l_2} z(\sigma) e^{-i\alpha \frac{\sigma}{L}} d\sigma, \qquad (94)$$

and

$$\frac{R^2 \alpha}{(L - R\alpha)} \simeq i \frac{\alpha}{L} \int_0^{l_1} z(\sigma) e^{i\alpha \frac{\sigma}{L}} d\sigma. \qquad (94a)$$

The factor $e^{i\theta}$ means the rotation of the whole outer terminal over the angle θ, the incomplete winding. If $\theta = 0$, then the two equations are complex conjugates. This means that the two terminals are the same except for a reflection with respect to the x axis.

The expression

$$\frac{R^2 \alpha}{[L - R\alpha]} \simeq i \frac{\alpha}{L} \int_0^{l_1} z(\sigma) e^{i\alpha \frac{\sigma}{L}} d\sigma$$

expanded in powers of $\frac{\alpha}{L}$, gives

$$R^2 \frac{\alpha}{L} \left\{ 1 + R\frac{\alpha}{L} - R^2 \frac{\alpha^2}{L^2} + \cdots \right\} \simeq i \frac{\alpha}{L} \int_0^{l_1} z(\sigma) \left\{ 1 + i\alpha \frac{\sigma}{L} - \frac{\alpha^2 \sigma^2}{2! L^2} + \cdots \right\}.$$

Now, if the coefficients of each power of α are equated, it follows that

Compared with (93)

$$\Delta\zeta = \frac{}{\alpha k_s}\left[\frac{}{[1-\alpha k]}\left[\varepsilon_1\varepsilon^{\,i\theta}\varepsilon^{\,i\alpha\frac{R}{k_s}} - \varepsilon^{\,i\alpha}\varepsilon^{\,-i\alpha\frac{R}{k_1}}\right]\right].$$

gives

$$\varepsilon^{\,i\theta}\frac{}{[1-k\alpha]}\frac{}{k_s\alpha} \simeq -i\frac{R}{\alpha}\int_0^{} s(\alpha)\varepsilon^{\,-i\alpha\frac{R}{\alpha}}\,d\alpha\,, \qquad (91m)$$

and

$$\frac{}{(1-k\alpha)}\frac{}{k_s\alpha} \simeq i\frac{R}{\alpha}\int_{k_1}^{} s(\alpha)\varepsilon^{\,i\alpha\frac{R}{\alpha}}\,d\alpha\,. \qquad (91a)$$

The factor $\varepsilon^{i\theta}$ means the rotation of the whole outer terminal over the angle θ ; the incomplete winding. If $\theta = 0$, then the two equations are complex conjugates. This means that the two terminals are the same except for a reflection with respect to the x axis.

The expression

$$\frac{}{[1-k\alpha]}\frac{}{k_s\alpha} \simeq i\frac{R}{\alpha}\int_{k_1}^{} s(\alpha)\varepsilon^{\,i\alpha\frac{R}{\alpha}}\,d\alpha$$

expanded in powers of $\frac{R}{\alpha}$, gives

$$\frac{R}{\alpha}\left\{1 + k\alpha + \frac{R}{\alpha}k_s\alpha^2 + \cdots\right\} \simeq i\frac{R}{\alpha}\int_0^{} s(\alpha)\left\{1 + i\alpha\frac{R}{\alpha} - \alpha^2\frac{R^2}{\alpha^2}s_2 + \cdots\right\}.$$

Now, if the coefficients of each power of α are equated, it follows that

$$R^2 = i\int_0^\ell z(\sigma)\,d\sigma = i\int_0^\ell (x+iy)\,d\sigma = \int_0^\ell (ix-y)\,d\sigma. \tag{95}$$

From this the conditions of Phillips can be obtained:

$$R^2 = -\int_0^\ell y\,d\sigma\;; \quad R^2 = -y\ell, \tag{95a}$$

and

$$0 = \int_0^\ell x\,d\sigma. \tag{95b}$$

This means that the terminal must have its center of gravity on the negative y axis, a distance $\frac{R^2}{\ell}$ from the origin.

$$R_s = c \int_0^\ell \varepsilon(\tau)\, d\tau = c \int_0^\ell (x + i\eta)\, d\tau = \int_0^\ell (\xi x - \eta)\, d\tau. \tag{95}$$

From this the conditions of Phillips can be obtained:

$$R_s = -\int_0^\ell \eta\, d\tau \; ; \quad R_s = -\lambda_\ell , \tag{95a}$$

and

$$0 = \int_0^\ell x\, d\tau. \tag{95b}$$

This means that the terminal must have its center of gravity on the negative y axis, a distance $\frac{R_s}{\ell}$ from the origin.

17. The Second Approximation

To investigate the next requirements the coefficients of $\left(\frac{\alpha}{\lambda}\right)^2$ are compared; hence,

$$R^3 = -\int_0^\ell z(\sigma)\,\sigma\,d\sigma\,, \qquad (96)$$

or

$$R^3 = -\int_0^\ell (x+iy)\,\sigma\,d\sigma\,;$$

whence,

$$R^3 = -\int_0^\ell x\,\sigma\,d\sigma\,,$$

and

$$0 = -\int_0^\ell y\,\sigma\,d\sigma\,.$$

The requirements for the spring are that its terminal fulfills both equations (95) and (96); moreover,

$$z(\sigma) = R \qquad \text{for} \qquad \sigma = 0\,.$$

This is so because the terminal has to fit onto the "helical" part of the spring. J. Haag*(3) states that he has been unable to find a solution; however, a very good solution is given by M.C. Wang.*(2)

17. The Second Approximation

To investigate the next requirements the coefficients of $\left(\frac{r}{a}\right)^s$ are compared; hence,

$$K_s = -\int_0^1 z(\alpha)\,\alpha\,d\alpha\,, \qquad (96)$$

or,

$$K_3 = -\int_0^1 (x+\lambda)\,\alpha\,d\alpha\,;$$

whence,

$$K_3 = -\int_0^1 x\,\alpha\,d\alpha\,,$$

and

$$0 = -\int_0^1 \lambda\,\alpha\,d\alpha\,.$$

The requirements for the spring are that its terminal fulfills both equations (95) and (96); moreover,

$$z(\alpha) = K \qquad \text{for} \qquad \alpha = 0\,.$$

This is so because the terminal has to fit onto the "helical" part of the spring. J. Haag*(3) states that he has been unable to find a solution; however, a very good solution is given by M.C. Wang.*(2)

18. Spring With Fixed End Point

The displacement of the end point of the free spring is a measure of the reaction forces set up in the end points of the actual hairspring fastened to a stud.

$z_a(s)$ represents the coordinates of a spring, and $\phi_a(s)$ the function for the change of angle of the actual spring after rotation of the balance wheel over an angle α. Now, the requirement is that the end point remain fixed; hence.

$$z_a(\bar{L}) = z(L) = \text{constant}, \tag{97}$$

or, in components,

$$x_a(\bar{L}) = x_a(0) + \int_0^{\bar{L}} ds \cos(\psi + \phi_a) = \text{constant}, \tag{97a}$$

and

$$y_a(\bar{L}) = y_a(0) + \int_0^{\bar{L}} ds \sin(\psi + \phi_a) = \text{constant}. \tag{97b}$$

The extreme value of the potential energy

$$V_a = K \int_0^{\bar{L}} (\phi_a')^2 ds$$

subject to these constraints must be determined. With the use of Lagrange multiplyers 2λ and 2μ, which are proportional to the x and y components of the reaction force caused by the constraint, the equations become

16. Spring With Fixed End Point

The displacement of the end point of the free spring is a measure of the reaction forces set up in the end points of the actual hairspring fastened to a stud.

$\xi_\alpha(s)$ represents the coordinates of a spring, and $\phi_\alpha(s)$ the function for the change of angle of the actual spring after rotation of the balance wheel over an angle α . Now, the requirement is that the end point remain fixed; hence.

$$\xi_\alpha(L) = \xi(L) = \text{constant}, \tag{37}$$

or, in components,

$$X_\alpha(L) = X_\alpha(0) + \int_0^L ds \cos(\psi + \phi_\alpha) = \text{constant} \tag{37a}$$

and

$$Y_\alpha(L) = Y_\alpha(0) + \int_0^L ds \sin(\psi + \phi_\alpha) = \text{constant}. \tag{37b}$$

The extreme value of the potential energy

$$V_\alpha = k \int_0^L (\phi'_\alpha)^2 ds$$

subject to these constraints must be determined. With the use of Lagrange multipliers 2λ and 2μ , which are proportional to the x and y components of the reaction force caused by the constraint, the equations become

$$x_a(\bar{L}) = x_a(0) + \int_0^{\bar{L}} ds \cos(\psi+\phi_a) \Big| 2\lambda = \text{constant},$$

and

$$y_a(\bar{L}) = y_a(0) + \int_0^{\bar{L}} ds \sin(\psi+\phi_a) \Big| 2\mu = \text{constant}.$$

Moreover, the change in curvature is

$$2\lambda \cos(\psi+\phi_a)\, ds + 2\mu \sin(\psi+\phi_a)\, ds.$$

If this expression is substituted into the potential energy equation, and minimized, it follows that

$$\delta V = \delta \int_0^{\bar{L}} \left[\phi_a'^2(s) + 2\lambda \cos(\psi+\phi_a) + 2\mu \sin(\psi+\phi_a) \right] ds = 0.$$

The Euler Lagrange equation now becomes

$$\frac{d}{ds}\left\{ \frac{d}{d\phi_a'} \left[\phi_a'^2 + 2\lambda \cos(\psi+\phi_a) + 2\mu \sin(\psi+\phi_a) \right] \right\} + \frac{d}{d\phi_a} \left[\phi_a'^2 + 2\lambda \cos(\psi+\phi_a) + 2\mu \sin(\psi+\phi_a) \right] = 0,$$

or

$$\phi''(s) + \lambda \sin(\psi+\phi_a) - \mu \cos(\psi+\phi_a) = 0. \tag{98}$$

This equation in general cannot be solved, but some interesting conclusions can be derived from it. If it is anticipated that the distribution will differ

little from that of the free end spring, it can be written that

$$\phi_a(s) = \phi(s) + \varepsilon(s) = -\frac{\alpha}{L}s + \alpha + \varepsilon(s),$$

where $\varepsilon(s)$ is a small correction term; moreover,

$$\phi_a'(s) = -\frac{\alpha}{L} + \varepsilon'(s),$$

and

$$\phi_a''(s) = \varepsilon''(s).$$

It was found that for any position of the spring there exists the relations

$$x_a'(s) = \cos(\psi + \phi_a),$$

and

$$y_a'(s) = \sin(\psi + \phi_a);$$

hence, the differential equation can be put into the form

$$\varepsilon''(s) + \lambda y_a' - \mu x_a' = 0 \quad , \text{ and} \qquad (99)$$

has boundary values

$$s = 0 \quad , \quad \varepsilon(0) = 0,$$

and

$$s = L \quad , \quad \varepsilon(L) = 0.$$

From this differential equation integrated once, results the expression

$$\varepsilon'(s) + \lambda y_a - \mu x_a = \text{constant}. \qquad (100)$$

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$$\phi_\alpha(s) = \phi_\alpha(s) + \varepsilon(s) = -\frac{\alpha}{L}s + \alpha + \varepsilon(s) ,$$

where $\varepsilon(s)$ is a small correction term; moreover,

$$\phi_\alpha'(s) = -\frac{\alpha}{L} + \varepsilon'(s) ,$$

and

$$\phi_\alpha''(s) = \varepsilon''(s) .$$

It was found that for any position of the spring there exists the relations

$$\chi_\alpha'(s) = \cos(\psi+\phi_\alpha) ,$$

and

$$\chi_\alpha'(s) = \sin(\psi+\phi_\alpha) ;$$

hence, the differential equation can be put into the form

$$\varepsilon''(s) + \lambda\chi_\alpha' - \mu\chi_\alpha' = 0 \quad , \text{ and} \qquad (99)$$

has boundary values

$$s = 0 \quad , \quad \varepsilon(0) = 0 ,$$

and

$$s = L \quad , \quad \varepsilon(L) = 0 .$$

From this differential equation integrated once, results the expression

$$\varepsilon'(s) + \lambda\chi_\alpha - \mu\chi_\alpha = \text{constant.} \qquad (100)$$

The constant is determined by a second integration with use of the limits 0 to $\bar{L}$, and the boundary conditions for $\varepsilon(s)$. This equation becomes

$$\varepsilon\Big|_0^{\bar{L}} + \lambda\int_0^{\bar{L}} y\,ds - \mu\int_0^{\bar{L}} x\,ds = c\,s\Big|_0^{\bar{L}},$$

or

$$\lambda\int_0^{\bar{L}} y\,ds - \mu\int_0^{\bar{L}} x\,ds = c\,\bar{L},$$

which can be written as

$$\lambda\frac{\bar{L}}{\bar{L}}\int_0^{\bar{L}} y\,ds - \mu\frac{\bar{L}}{\bar{L}}\int_0^{\bar{L}} x\,ds = c\,\bar{L}. \qquad (101)$$

With

$$\frac{1}{\bar{L}}\int_0^{\bar{L}} y\,ds = Y_a, \qquad (102)$$

and

$$\frac{1}{\bar{L}}\int_0^{\bar{L}} x\,ds = X_a, \qquad (102a)$$

the coordinates of the center of gravity of the distorted spring, equation (101) becomes

$$\lambda Y_a - \mu X_a = c. \qquad (103)$$

This expression for C, replaced in

$$\varepsilon'(s) + \lambda y_a - \mu x_a = C,$$

gives the final result

The constant is determined by a second integration with use of the limits 0 to L, and the boundary conditions for $\varepsilon(s)$. This equation becomes

$$\varepsilon \Big|_0^L + \gamma \int_0^L \lambda\, ds - M \int_0^L x\, ds = c\, s \Big|_0^L ,$$

or

$$\gamma \int_0^L \lambda\, ds - M \int_0^L x\, ds = c\, L ,$$

which can be written as

$$\gamma \frac{L}{L} \int_0^L \lambda\, ds - M \frac{L}{L} \int_0^L x\, ds = c\, L . \qquad (101)$$

With

$$\frac{1}{L} \int_0^L \lambda\, ds = Y_a ; \qquad (102)$$

and

$$\frac{1}{L} \int_0^L x\, ds = X_a , \qquad (102a)$$

the coordinates of the center of gravity of the distorted spring, equation (101) becomes

$$\gamma Y_a - M X_a = C . \qquad (103)$$

This expression for C, replaced in

$$\varepsilon'(s) + \gamma \lambda - M x_a = C ,$$

gives the final result

$$\varepsilon'(s) = -\lambda(y_a - Y_a) + \mu(x_a - X_a) . \tag{104}$$

Now, the displacement of the end point Δ is represented by

$$\Delta = z_{free}(\bar{L}) - z_a(\bar{L}) = \int_0^{\bar{L}} (e^{i\psi_f} - e^{i\psi_a})\, ds .$$

Since $\varepsilon = \psi_a - \psi_f$,

$$\Delta = \int_0^{\bar{L}} (e^{-i(\varepsilon + \psi_a)} e^{i\psi_a} - e^{i\psi_a})\, ds = \int_0^{\bar{L}} e^{i\psi_a}(e^{-i\varepsilon} - 1)\, ds . \tag{105}$$

If ε is considered small, $e^{-i\varepsilon}$ can be expanded; moreover, the displacement, neglecting second order terms, becomes

$$\Delta = \int_0^{\bar{L}} ds\, e^{i\psi_a}[1 - i\varepsilon - 1] = -i\int_0^{\bar{L}} \varepsilon\, ds\, e^{i\psi_a} .$$

However,

$$d z_a(s) = ds\, e^{i\psi_a} ,$$

and

$$\Delta = -i \int_0^{\bar{L}} d z_a(s)\, \varepsilon . \tag{106}$$

$$\Delta = -\epsilon \int_0^{\bar{L}} d\xi_\alpha(s)\, \xi \,. \tag{105}$$

and

$$d\xi_\alpha(s) = ds\, e^{i\Phi_\alpha} \,.$$

However,

$$\Delta = -\epsilon \int_0^{\bar{L}} ds\, e^{i\Phi_\alpha} [1-\epsilon\xi^{-1}] = -\epsilon \int_0^{\bar{L}} \xi\, ds\, e^{i\Phi_\alpha} \,.$$

Now, the displacement of the end point Δ is represented by

$$\Delta = \xi(\bar{L}) - \xi_\alpha(\bar{L}) = \int_0^{\bar{L}} (e^{i\Phi} - e^{i\Phi_\alpha})\, ds \,;$$

since $\xi = \Phi_\alpha - \Phi$,

$$\Delta = \int_0^{\bar{L}} (e^{-\epsilon(\xi+\Phi_\alpha)} e^{i\Phi_\alpha} - e^{i\Phi_\alpha})\, ds = \int_0^{\bar{L}} e^{i\Phi_\alpha} (e^{-\epsilon\xi} - 1)\, ds \tag{105}$$

If ξ is considered small, $e^{-\epsilon\xi}$ can be expanded; moreover, the displacement, neglecting second order terms, becomes

$$\Delta = \int_0^{\bar{L}} ds\, e^{i\Phi_\alpha} [1-\epsilon\xi-1] = -\epsilon \int_0^{\bar{L}} \xi\, ds\, e^{i\Phi_\alpha} \,.$$

However,

$$d\xi_\alpha(s) = ds\, e^{i\Phi_\alpha} \,,$$

and

$$\Delta = -\epsilon \int_0^{\bar{L}} d\xi_\alpha(s)\, \xi \,. \tag{106}

From equation (106), integrated by parts, the following equation can be obtained:

$$\Delta \simeq -i\varepsilon z\Big|_0^{\bar{L}} + i\int_0^{\bar{L}} z\varepsilon' ds \simeq i\int_0^{\bar{L}} z\varepsilon' ds, \tag{107}$$

and it follows that upon substitution of ε' that

$$\Delta \simeq i\int_0^{\bar{L}} \bar{z}_a(s)\,\varepsilon' ds = i\int_0^{\bar{L}} \bar{z}_a(s)\left\{-\lambda(y_a-\bar{Y}_a)+\mu(x_a-\bar{X}_a)\right\} ds. \tag{108}$$

This equation expressed in components becomes

$$\Delta x + i\Delta y = \int_0^{\bar{L}} \Big[i x\left\{-\lambda(y_a-\bar{Y}_a)+\mu(x_a-\bar{X}_a)\right\} - y\left\{-\lambda(y_a-\bar{Y}_a)+\mu(x_a-\bar{X}_a)\right\}\Big] ds, \tag{109}$$

where

$$\Delta x = \int_0^{\bar{L}} \Big[\lambda(y_a^2 - \bar{Y}_a y_a) - \mu(x_a y_a - \bar{X}_a y_a)\Big] ds, \tag{109a}$$

and

$$\Delta y = \int_0^{\bar{L}} \Big[-\lambda(x_a y_a - x_a \bar{Y}_a) + \mu(x_a^2 - x_a \bar{X}_a)\Big] ds. \tag{109b}$$

It follows that

$$\Delta x = \lambda\int_0^{\bar{L}} y_a^2 ds - \lambda\int_0^{\bar{L}} \bar{Y}_a y_a ds - \mu\int_0^{\bar{L}} x_a y_a ds + \mu\int_0^{\bar{L}} \bar{X}_a y_a ds, \tag{110}$$

From equation (106), integrated by parts, the following equation can be obtained:

$$\Delta \simeq -\varepsilon' \varepsilon \int_0^{\bar{L}} + \varepsilon \int_0^{\bar{L}} \varepsilon \varepsilon' \, dz \simeq \varepsilon \int_0^{\bar{L}} \varepsilon \varepsilon' \, dz \,, \tag{107}$$

and it follows that upon substitution of ε' that

$$\Delta \simeq \varepsilon \int_0^{\bar{L}} \frac{\varepsilon_a(s)}{\varepsilon_a(s)} \varepsilon' \, dz = \varepsilon \int_0^{\bar{L}} \frac{\varepsilon_a(s)}{\varepsilon_a(s)} \left\{ -\gamma(\chi_a - \chi_a') + \mu(\chi_a - \chi_a') \right\} dz \,. \tag{108}$$

This equation expressed in components becomes

$$\Delta x + \varepsilon \Delta \gamma = \varepsilon \int_0^{\bar{L}} \left[\varepsilon x \left\{ -\gamma(\chi_a - \chi_a') + \mu(\chi_a - \chi_a') \right\} \right.$$

$$\left. - \gamma \left\{ -\gamma(\chi_a - \chi_a') + \mu(\chi_a - \chi_a') \right\} \right] dz \,, \tag{109}$$

where

$$\Delta x = \int_0^{\bar{L}} \left[\gamma(\chi_a' \chi_a - \chi_a' \chi_a) - \mu(\chi_a \chi_a - \chi_a \chi_a') \right] dz \,, \tag{109a}$$

and

$$\Delta \gamma = \int_0^{\bar{L}} \left[-\gamma(\chi_a \chi_a - \chi_a \chi_a') + \mu(\chi_a' - \chi_a \chi_a') \right] dz \,. \tag{109b}$$

It follows that

$$\Delta x = \gamma \int_0^{\bar{L}} \chi_a' \, dz - \gamma \int_0^{\bar{L}} \chi_a \chi_a \, dz - \mu \int_0^{\bar{L}} \chi_a \chi_a \, dz + \mu \int_0^{\bar{L}} \chi_a \chi_a \, dz \,, \tag{110}$$

and

$$\Delta y = -\lambda \int_0^{\bar{L}} x_a y_a \, ds + \lambda \int_0^{\bar{L}} x_a \bar{Y}_a \, ds + \mu \int_0^{\bar{L}} x_a^2 \, ds - \mu \int_0^{\bar{L}} x_a \bar{X}_a \, ds,$$

or

$$\Delta x = \lambda \int_0^{\bar{L}} y_a^2 \, ds - \lambda \bar{Y}_a \bar{Y}_a \bar{L} - \mu \int_0^{\bar{L}} x_a y_a \, ds + \mu \bar{X}_a \bar{Y}_a \bar{L}, \quad (110a)$$

and

$$\Delta y = -\lambda \int_0^{\bar{L}} x_a y_a \, ds + \lambda \bar{Y}_a \bar{X}_a \bar{L} + \mu \int_0^{\bar{L}} x_a^2 \, ds - \mu \bar{X}_a^2 \bar{L}; \quad (110b)$$

where,

$$\bar{L} A = \int_0^{\bar{L}} y_a^2 \, ds - \bar{L} \bar{Y}_a^2,$$

$$\bar{L} B = \int_0^{\bar{L}} x_a^2 \, ds - \bar{L} \bar{X}_a^2, \quad (111)$$

and

$$\bar{L} F = \int_0^{\bar{L}} x_a y_a \, ds - \bar{L} \bar{X}_a \bar{Y}_a .$$

Equations (110a) and (110b) are now

$$\Delta x = \lambda \bar{L} A - \mu \bar{L} F, \quad (112)$$

and

$$\Delta y = -\lambda F \bar{L} + \mu \bar{L} B. \quad (112a)$$

A, B, and F are the moments and products of inertia with respect to the center of gravity of the distorted

and

$$\Delta\lambda = -\lambda \int_0^{\bar{L}} x_\sigma y_\sigma \, ds + \lambda \int_0^{\bar{L}} \left(x_\sigma Y_\sigma \, ds + \mu \int_0^{\bar{L}} x_\sigma^2 \, ds - \mu \int_0^{\bar{L}} x_\sigma X_\sigma \, ds \right),$$

or

$$\Delta x = \lambda \int_0^{\bar{L}} x_\sigma^2 \, ds - \lambda \bar{Y}_\sigma \bar{L} - \mu \int_0^{\bar{L}} x_\sigma y_\sigma \, ds + \mu \bar{X}_\sigma \bar{Y}_\sigma \bar{L}^2 \quad (110a)$$

and

$$\Delta\lambda = -\lambda \int_0^{\bar{L}} x_\sigma y_\sigma \, ds + \lambda \bar{X}_\sigma \bar{L} + \mu \int_0^{\bar{L}} x_\sigma^2 \, ds - \mu \bar{X}_\sigma^2 \bar{L}^2 \quad (110b)$$

where

$$\bar{L}\, A = \int_0^{\bar{L}} x_\sigma^2 \, ds - \bar{L}\, \bar{X}_\sigma^2,$$

$$\bar{L}\, B = \int_0^{\bar{L}} x_\sigma^2 \, ds - \bar{L}\, \bar{X}_\sigma^2, \quad (111)$$

and

$$\bar{L}\, F = \int_0^{\bar{L}} x_\sigma y_\sigma \, ds - \bar{L}\, \bar{X}_\sigma \bar{Y}_\sigma .$$

Equations (110a) and (110b) are now

$$\Delta x = \lambda \bar{L} A - \mu \bar{L} F, \quad (112)$$

and

$$\Delta\lambda = -\lambda F \bar{L} + \mu \bar{L} B. \quad (112a)$$

A, B, and F are the moments and products of inertia with respect to the center of gravity of the distorted

spring. These two equations solved for λ and μ give

$$\lambda = \frac{(B\Delta x + F\Delta y)}{L(AB - F^2)}, \tag{113}$$

and

$$\mu = \frac{(F\Delta x + A\Delta y)}{L(AB - F^2)}.$$

The square of the reaction force is obtained by

$$\lambda^2 + \mu^2 = \frac{(B\Delta x + F\Delta y)^2 + (F\Delta x + A\Delta y)^2}{L^2(AB - F^2)^2}. \tag{114}$$

Usually $A \simeq B$, and $F = 0$ so that a first approximation gives

$$\lambda^2 + \mu^2 \simeq \frac{|\Delta|^2}{A^2 L^2}.$$

This result proves that the reaction force is approximately proportional to the displacement.

spring. These two equations solved for λ and μ

give

$$\lambda = \frac{(B\Delta x + F\Delta y)}{\Gamma_M (AB - F_c^2)} , \qquad (113)$$

and

$$\mu = \frac{(F\Delta x + A\Delta y)}{\Gamma_M (AB - F_c^2)} .$$

The square of the reaction force is obtained by

$$\lambda^2 + \mu^2 = \frac{(B\Delta x + F\Delta y)^2 + (F\Delta x + A\Delta y)^2}{\Gamma_M^2 (AB - F_c^2)^2} \qquad (114)$$

Usually $A \doteq B$, and $F = 0$ so that a first approximation

gives

$$\lambda_c^2 + \mu_c^2 \leq \frac{|\Delta|^2}{A^2 \Gamma_M^2} .$$

This result proves that the reaction force is

approximately proportional to the displacement.

19. Circular Spring with Fixed End Point

A spring, characterized by a circle, has its moment of inertia A with respect to the x axis equal to the moment of inertia B with respect to the y axis; however, the moment of inertia C along the z axis and the product of inertia F are zero.

The moment of inertia of the spring with respect to the origin now becomes

$$A + B = C,$$

where

$$C = 2A = 2B.$$

The reaction force for the spring can now be written as

$$\sqrt{\lambda^2+\mu^2} = \sqrt{(B\Delta x)^2+(A\Delta y)^2} = \frac{|\Delta|}{A\overline{L}} = 2\frac{|\Delta|}{C\overline{L}}. \quad (115)$$

$$\overline{L}\,(AB)$$

19. Circular Spring with Fixed End Point

A spring, characterized by a circle, has its moment of inertia A with respect to the x axis equal to the moment of inertia B with respect to the y axis; however, the moment of inertia C along the $\frac{\pi}{4}$ axis and the product of inertia F are zero.

The moment of inertia of the spring with respect to the origin now becomes

$$A + B = C ,$$

where

$$C = 2A = 2B .$$

The reaction force for the spring can now be written as

$$\sqrt{Y_s^2 + M_s^2} = \sqrt{(B\Delta x)^2 + (A\Delta x)^2} = \frac{|\Delta|}{L(AB)} \frac{AL}{CL} = 2|\Delta| . \quad (115)$$

20. Elastic Energy for Spring with Fixed End Point

The energy[*(2)] of the distorted spring with a fixed end point is

$$V = \frac{\xi}{2}\int_0^{\bar{L}} \phi_a'^2\, ds = \frac{\xi}{2}\int_0^{\bar{L}} (\phi' + \varepsilon')\, ds = \frac{\xi}{2}\frac{\alpha^2}{\bar{L}} + \frac{\xi}{2}\int_0^{\bar{L}} \varepsilon'^2\, ds .$$

For the value of ε' substituted into the above equation, the energy can be expressed as

$$V = \frac{\xi}{2}\frac{\alpha^2}{\bar{L}} + \frac{\xi}{2}\int_0^{\bar{L}} \left[-\lambda(y_a - \bar{Y}_a) + \mu(x_a - \bar{X}_a)\right]^2 ds ,$$

or

$$V = \frac{\xi}{2\bar{L}}\alpha^2 + \frac{\xi}{2}\left\{\int_0^{\bar{L}} \lambda^2 (y_a - \bar{Y}_a)^2 ds + \int_0^{\bar{L}} \mu (x_a - \bar{X}_a)^2 ds - 2\lambda\mu \int_0^{\bar{L}} (y_a - \bar{Y}_a)(x_a - \bar{X}_a)\, ds\right\};$$

hence,

$$V = \frac{\xi}{2\bar{L}}\alpha^2 + \frac{\xi}{\bar{L}}\left\{\lambda^2\left[\int_0^{\bar{L}} y_a^2 ds - 2\int_0^{\bar{L}} \bar{Y}_a y_a\, ds + \int_0^{\bar{L}} \bar{Y}_a^2 ds\right] + \mu^2\left[\int_0^{\bar{L}} x_a^2 ds - 2\int_0^{\bar{L}} \bar{X}_a x_a\, ds + \int_0^{\bar{L}} \bar{X}_a^2 ds\right] - 2\lambda\mu\left[\int_0^{\bar{L}} x_a y_a\, ds - \int_0^{\bar{L}} x_a \bar{Y}_a\, ds - \int_0^{\bar{L}} \bar{X}_a y_a\, ds + \int_0^{\bar{L}} \bar{Y}_a \bar{X}_a\, ds\right]\right\}.$$

20. Elastic Energy for Spring with Fixed End Point

The energy[2] of the distorted spring with a fixed end point is

$$V = \frac{mE}{2}\int_0^L \phi'^2 ds = \frac{S}{2E}\int_0^L (\phi+\varepsilon')^2 ds = \frac{\alpha_S}{2}\frac{S}{E} + \frac{S}{2E}\int_0^L \varepsilon'^2 ds .$$

For the value of ε' substituted into the above equation, the energy can be expressed as

$$V = \frac{\alpha_S}{2}\frac{S}{E} + \frac{S}{2E}\int_0^L \left[-\gamma(\lambda_\sigma - \Lambda_\sigma) + \mu(x_\sigma - X_\sigma)\right]^2 ds ,$$

or

$$V = \frac{\alpha_S}{2}\frac{S}{E} + \frac{S}{2E}\left\{\int_0^L \gamma^2(\lambda_\sigma - \Lambda_\sigma)^2 ds + \int_0^L \mu^2(x_\sigma - X_\sigma)^2 ds\right.$$

$$\left. - 2\gamma\mu\int_0^L (\lambda_\sigma - \Lambda_\sigma)(x_\sigma - X_\sigma) ds\right\} .$$

Hence,

$$V = \frac{\alpha_S}{2}\frac{S}{E} + \frac{S}{2E}\left\{\gamma^2\left[\int_0^L \lambda_\sigma^2 ds - 2\int_0^L \lambda_\sigma \Lambda_\sigma ds + \int_0^L \Lambda_\sigma^2 ds\right]\right.$$

$$+\mu^2\left[\int_0^L x_\sigma^2 ds - 2\int_0^L x_\sigma X_\sigma ds + \int_0^L X_\sigma^2 ds\right]$$

$$\left. - 2\gamma\mu\left[\int_0^L \lambda_\sigma x_\sigma ds - \int_0^L \lambda_\sigma X_\sigma ds - \int_0^L x_\sigma \Lambda_\sigma ds + \int_0^L \Lambda_\sigma X_\sigma ds\right]\right\} .$$

With (102a) and (102b),

$$V = \frac{\xi}{2\bar{L}}\alpha^2 + \frac{\xi}{2}\left\{\lambda^2\left[\int_0^{\bar{L}} y_a^2\,ds - 2\bar{L}\bar{Y}_a^2 + \bar{Y}_a^2\bar{L}\right]\right.$$

$$+\mu^2\left[\int_0^{\bar{L}} x_a^2\,ds - 2\bar{X}_a^2\bar{L} + \bar{X}_a^2\bar{L}\right]$$

$$\left.-2\mu\lambda\left[\int_0^{\bar{L}} x_a y_a\,ds - \bar{X}_a\bar{Y}_a\bar{L} - \bar{X}_a\bar{Y}_a\bar{L} + \bar{Y}_a\bar{X}_a\bar{L}\right]\right\}$$

becomes

$$V = \frac{\xi}{2\bar{L}}\alpha^2 + \frac{\xi}{2}\left\{\lambda^2\bar{L}A + \mu^2\bar{L}B - 2\mu\lambda\bar{L}F\right\}.$$

With the use of equations (113) the energy expression takes the form

$$V = \frac{\xi}{2\bar{L}}\alpha^2 + \frac{\xi}{2}\left\{\lambda(\lambda\bar{L}A - \mu\bar{L}F) + \mu(\mu\bar{L}B - \lambda\bar{L}F)\right\}.$$

The values of equations (112a) and (112b) substituted into

$$V = \frac{\xi}{2\bar{L}}\alpha^2 + \frac{\xi}{2}\left\{\lambda\Delta x + \mu\Delta y\right\}$$

alters the form of the energy equation to

$$V = \frac{\xi}{2\bar{L}}\alpha^2 + \frac{\xi}{2}\left\{\frac{B\Delta x^2 + A\Delta y^2 + 2F\Delta x\Delta y}{\bar{L}(AB - F^2)}\right\}. \qquad (116)$$

$$V = \sum_{s=1}^{n} \alpha_s + \sum_{s=1}^{n} \left\{ \frac{B\Delta x_s + A\Delta y_s + 2F\Delta x\Delta y}{L(AB - F_s)} \right\} . \qquad (112)$$

Since $A \simeq B$, and $F = 0$, the energy equation is

$$V \simeq \frac{\varepsilon}{2L}\alpha^2 + \frac{\varepsilon}{2}\frac{|\Delta|^2}{LA} = \frac{\varepsilon}{2L}\left(\alpha^2 + \eta(\alpha)\right). \qquad (117)$$

The correction term, abreviated $\eta(\alpha)$ causes the deviation from isochronism.

Since $A \neq 0$, and $\Gamma = 0$, the energy equation is

$$V_{\text{ac}} \underline{\underline{\varepsilon}} \alpha_s + \frac{\underline{\underline{\beta}}}{2} \frac{|A|^2}{\underline{\underline{L}} A} = \frac{\underline{\underline{\beta}}}{\underline{\underline{L}}} \cdot (\alpha_s + \eta(\alpha)) \cdot \qquad (117)$$

The correction term, abbreviated $\eta(\alpha)$ causes the deviation from isochronism.

21. Deviation from Isochronism

The balance wheel is much heavier than the hairspring, and it is responsible for the kinetic energy:

$$\frac{1}{2} I \left(\frac{d\alpha}{dt} \right)^2 .$$

Hence, the equation of motion of the balance wheel becomes:

$$\frac{d^2\alpha}{dt^2} = -\frac{\xi}{I} \left\{ 2\alpha + d\frac{\eta(\alpha)}{d\alpha} \right\},$$

or

$$\frac{d^2\alpha}{dt^2} = -\omega^2 \left\{ \alpha + \frac{1}{2} \frac{d\eta(\alpha)}{d\alpha} \right\}, \qquad (118)$$

where 2/T. If the correction term $\eta(\alpha)$ is not present, T is the period.

With $\alpha \simeq \alpha_0 \sin \omega(1+\delta)t \cdots$, it follows that

$$-\alpha_0 \left[\omega^2 (1+\delta)^2 \sin \omega(1+\delta)t = -\omega^2 \left\{ \alpha_0 \sin \omega(1+\delta)t + \frac{1}{2} \frac{d\eta(\alpha)}{d\alpha} \right. \right.,$$

or

$$\alpha_0 \left[(1+\delta)^2 - 1 \right] \sin \omega(1+\delta)t = \tfrac{1}{2} \frac{d\eta(\alpha)}{d\alpha} .$$

This equation multiplied by

$$\alpha \sin \omega(1+\delta)t$$

21. Deviation from Isochronism

The balance wheel is much heavier than the hairspring, and it is responsible for the kinetic energy:

$$\frac{1}{2} I \left(\frac{d\alpha}{dt} \right)^2 .$$

Hence, the equation of motion of the balance wheel becomes:

$$\frac{d}{dt} I \frac{d\alpha}{dt} = -\frac{E}{\ell} \left\{ \ell\alpha + d \frac{M(\alpha)}{d\alpha} \right\} ,$$

or

$$\frac{d^2\alpha}{dt^2} = -\omega_s^2 \left\{ \alpha + \frac{\ell}{2} \frac{d M(\alpha)}{d\alpha} \right\} , \qquad (118)$$

where ℓ/I. If the correction term $M(\alpha)$ is not present, T is the period.

With $\alpha = \alpha_0 \sin \omega(1+\varepsilon)t$ ---, it follows that

$$-\alpha_0 \left[\omega_s^2 (1+\varepsilon)^2 \sin \omega(1+\varepsilon)t = -\omega_s^2 \left\{ \alpha_0 \sin \omega(1+\varepsilon)t + \frac{\ell}{2} \frac{d M(\alpha)}{d\alpha} \right. \right.$$

or

$$\alpha_0 \left[(1+\varepsilon)^2 - 1 \right] \sin \omega(1+\varepsilon)t = \frac{\ell}{2} \frac{d M(\alpha)}{d\alpha} .$$

This equation multiplied by

$$\alpha \sin \omega(1+\varepsilon)t$$

gives

$$\alpha_0^2\left[(1+\delta)^2-1\right]\sin^2\omega(1+\delta)t = \tfrac{1}{2}\frac{d\eta(\alpha)}{d\alpha}\alpha .$$

Upon integration, it follows that

$$\alpha_0^2\left[2\delta-\delta^2\right]\int_0^T \frac{1}{2}(1-\cos 2\omega(1+\delta)t)\,dt = \frac{1}{2}\int_0^T \frac{d\eta(\alpha)}{d\alpha}\alpha\,dt ,$$

or

$$\alpha_0^2\left[2\delta-\delta^2\right]\left[t-\frac{\sin 2\omega(1+\delta)t}{2\omega(1+\delta)}\right]_0^T = \int_0^T \frac{d\eta(\alpha)}{d\alpha}\alpha\,dt .$$

If the second order terms are neglected, the first approximation becomes

$$2\delta\alpha_0^2 T = \int_0^T \frac{d\eta(\alpha)}{d\alpha}\alpha\,dt .$$

The fractional increase in frequency is

$$\delta \sim \frac{1}{\alpha_0^2 T}\int_0^T \frac{d\eta(\alpha)}{d\alpha}\alpha\,dt, \quad \text{or} \tag{119}$$

from

$$\xi\frac{|\Delta|^2}{2L A} = \xi\frac{\eta(\alpha)}{2L},$$

the equation (119) becomes

$$\delta \simeq \frac{2}{\alpha_0^2 T A}\int_0^T |\Delta|\frac{d|\Delta|}{d\alpha}\alpha\,dt .$$

If $\eta(\alpha)$ is an odd function of α, the integral vanishes. However, $\eta(\alpha)$ is proportional to the square of $|\Delta(\alpha)|$ and can never be a pure odd function of α.

gives

$$\alpha_s^0\left[(1+\varepsilon)^{s}-1\right]\sin\omega_s(1+\varepsilon)t = \tfrac{1}{2}\frac{d\,\mathcal{M}(\alpha)}{d\alpha}\,\alpha\,.$$

Upon integration, it follows that

$$\alpha_s^0\left[\sin 2\theta - \varepsilon_s\right]\int_0^{\frac{s}{T}}(1-\cos 2\omega_s(1+\varepsilon)t)\,dt = \int_0^T \frac{d\,\mathcal{M}(\alpha)}{d\alpha}\,\alpha\,dt$$

or

$$\alpha_s^0\left[\sin 2\theta - \varepsilon_s\right]\left[t - \frac{\sin 2\omega_s(1+\varepsilon)t}{2\omega_s(1+\varepsilon)}\right]_0^{\frac{s}{T}} = \int_0^T \frac{d\,\mathcal{M}(\alpha)}{d\alpha}\,\alpha\,dt\,.$$

If the second order terms are neglected, the first approximation becomes

$$2\,\alpha_s^0 T = \int_0^T \frac{d\,\mathcal{M}(\alpha)}{d\alpha}\,\alpha\,dt\,.$$

The fractional increase in frequency is

$$\varepsilon_s = \frac{\alpha_s^0}{4T}\int_0^T \frac{d\,\mathcal{M}(\alpha)}{d\alpha}\,\alpha\,dt\,, \quad \text{or} \tag{119}$$

from

$$\frac{\Sigma\,|\Delta|_s}{\Sigma A} = \frac{\partial\,\mathcal{M}(\alpha)}{\partial\,\Sigma}\,,$$

the equation (119) becomes

$$\varepsilon_s = \frac{\alpha_s^0}{4TA}\int_0^T \frac{d\,|\Delta|}{d\alpha}\,\alpha\,dt\,.$$

If $\mathcal{M}(\alpha)$ is an odd function of α, the integral vanishes. However, $\mathcal{M}(\alpha)$ is proportional to the square of $|\Delta(\alpha)|$ and can never be a pure odd function of α.

If the even part of η is α^2 only, δ is some constant value independent of the amplitude α_0. Therefore, if $|\Delta|^2/\alpha^2$ is a constant plus an odd function of α, the period will change by a constant amount. In the case of the helical spring

$$\Delta \sim \alpha (e^{i\theta} - e^{i\alpha});$$

which gives

$$|\Delta|^2/\alpha^2 \sim [1 - \cos(\alpha - \theta)]$$

when $\theta = \pm \pi/2$, $\cos(\alpha - \theta) = \pm \sin \alpha$, and $|\Delta|^2/\alpha^2$ fulfills the requirement for constant change of period, which proves the theorem of Resal.

Because $\eta(\alpha)$ is proportional to $|\Delta|^2$, the integral representing δ is small if $|\Delta|$ is small, provided, however, that $\frac{d|\Delta|}{d\alpha}$ does not become too large. This confirms that the best way to achieve isochronism is to make the end point displacement as small as possible. This result can be connected with the motion of the center of gravity of the free spring. With the use of (76) and (79), can be obtained

$$\frac{d\Delta}{d\alpha} = iz(0)e^{i\alpha} + i\int_0^{\bar{L}} e^{i\psi_f} ds - \frac{i}{\bar{L}}\int_0^{\bar{L}} s e^{i\psi_f} ds, \qquad (120)$$

or

$$\frac{d\Delta}{d\alpha} = \frac{i}{\bar{L}}\int_0^{\bar{L}} z_f(s)\, ds = iG.$$

If the even part of ψ is α_s only, σ is some constant value independent of the amplitude α_0. Therefore, if $|\Delta|^s/\alpha_s$ is a constant plus an odd function of α, the period will change by a constant amount. In the case of the helical spring

$$\Delta = \alpha\,(\epsilon_{\theta} - \epsilon_{\alpha})\,;$$

which gives

$$|\Delta|^s/\alpha_s = \sigma\left[1 - \cos(\alpha - \theta)\right]$$

when $\theta = \pm\,\pi/2$, $\cos(\alpha - \theta) = \pm \sin\alpha$, and $|\Delta|^s/\alpha_s$ fulfills the requirement for constant change of period, which proves the theorem of Resal.

Because $\psi(\alpha)$ is proportional to $|\Delta|^s$, the integral representing σ is small if $|\Delta|$ is small, provided, however, that $\frac{d\,|\Delta|}{d\,\alpha}$ does not become too large. This confirms that the best way to achieve isochronism is to make the end point displacement as small as possible. This result can be connected with the motion of the center of gravity of the free spring. With the use of (76) and (79), can be obtained

$$\frac{d\,\sigma}{d\,\alpha} = \epsilon_1 s(o)\, e^{i\alpha} + \epsilon_1 \int_0^{\bar{L}} \left(e^{i\psi} ds - \frac{\bar{L}}{\epsilon}\right) sc\, e^{i\psi} ds\,, \tag{120}$$

or

$$\frac{d\,\sigma}{\epsilon_1} = \int_0^{\bar{L}} s^t(s)\, ds = \epsilon_1 G\,.$$

G is the complex coordinate of the center of gravity of the free spring and changes its position with different rotations. Equation (119) can now be written as

$$\delta = \frac{2}{\alpha_0^2 \bar{T} A} \int_0^T \left[\bar{G} \times \bar{\Delta} \right] \alpha \, dt .$$

It is advantageous to have G as small as possible for all values of α. To have the center of gravity of the undistorted spring ($\alpha = 0$) fall exactly on the axis is, however, not necessarily the best solution (except for small amplitudes), because G is a complicated function of α.

Ω is the complex coordinate of the center of gravity of the free spring and changes its position with different rotations. Equation (11) can now be written as

$$\zeta = \frac{m}{\alpha_0^2 TA} \int_0^T [\bar{G} \times \bar{\Delta}] \alpha \, dt .$$

It is advantageous to have Ω as small as possible for all values of α. To have the center of gravity of the undistorted spring ($\alpha = 0$) fall exactly on the axis is, however, not necessarily the best solution (except for small amplitudes), because Ω is a complicated function of α.

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Abstract

The system of the hairspring and balance wheel used to regulate watches and chronometers does not perform an exact simple harmonic motion. For the proper function of the watch the deviation from isochronism of the vibrations, caused by the reaction force at the fixed end point for any distortion of the spring, must be as small as possible. This reaction force is found to be correspondingly small if the shape of the hairspring is designed so that the reaction force at the end point is small.

In theory only, perfect isochronism can be accomplished by a flat spring having one end free so that for any distortion of the spring, the tangent to the end point moves parallel to itself; however, the free end spring cannot be realized in a watch.

Phillips' solution to the problem primarily expresses that the reaction force at the end point is proportional to the displacement of the end point that would occur for a free spring. His solution is based on the problem of finding a possible curve such that the terminal point has a given trajectory or displacement, and simultaneously, at any point of the trajectory the tangent has an imposed direction.

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Phillips' solution to the problem primarily expresses that the reaction force at the end point is proportional to the displacement of the end point that would occur for a free spring. His solution is based on the problem of finding a possible curve such that the terminal point has a given trajectory or displacement, and simultaneously, at any point of the trajectory the tangent has an imposed direction.

For a spring having the outer end point fastened to a

stud, and the inner end point fastened to a balance wheel, Phillips requires that the distortion of any element of the spring satisfy the Bernoulli Euler Law

$$\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) = \frac{M}{EI} = \beta \text{ , a constant,}$$

where M, E, and I represent respectively, the bending moment, Youngs Modulus, and the moment of inertia of the spring. If the center of the balance wheel is taken at the origin of a coordinate system x, y, the radius of curvature at any arbitrary point of the spring makes an angle θ_0 and θ for the spring in the non-deformed and deformed state.

In the first method of solution Phillips considers the radius r of the balance wheel to be constant. In the initial state, the tangent to an arbitrary point along the spring, a function of the arc length, s, makes an angle θ_0 with the y axis. Then for the deformed state the Bernoulli Euler equation becomes $\left(\frac{\theta}{s} - \frac{\theta_0}{s}\right) = -\beta$, or $\theta = \theta_0 - \beta S$. The constant $\frac{M}{EI} = \beta$ is chosen negative because the angle θ_0 decreases as the spring increases. The angle which the tangent to the inner end point makes with the y axis is Φ_0 , for the initial state, and for the deformed state this angle becomes $\Phi = \Phi_0 - \beta L$, where L is the length of the spring. The conditions that the rotations $\Phi = \Phi_0 - \beta L$ and $\theta = \theta_0 - \beta S$ be satisfied identically, require that

the rotation $\Theta = \Theta_0 - \beta L$ of the radius r be very small. This condition signifies that the center of gravity of the spring coincides with the axis of the balance wheel.

In the second method of solution, Phillips assumes that the radius r, to which the inner end point is fastened, varies according to the law $\left(\frac{1}{r} - \frac{1}{r_0}\right) = \beta$. Imposing on this problem the same conditions of the first problem, the result requires that the center of gravity of the spring be on a line through the origin, perpendicular to the radius r in the initial state. The distance of the center of gravity from the origin satisfies the relation $\frac{r_0^2}{L}$.

The second problem of Phillips permits the designing of the fastened terminals of a helix so that the reaction force at the end points is small.

Resal found that it is possible to design the shape of the hairspring so that the reaction force changes the period of vibrations by an amount which is approximately independent of the amplitude. For this solution the reaction force need not be small.

The shape of the spring can be characterized by giving at each point the angle $\psi_{(s)}$ which the tangent makes with the x axis as a function of s, measured from the inner end point.

When the balance wheel is rotated over an angle α, the spring will be distorted and the new shape can be characterized by giving the change in $\psi_{(s)}$. The change of angle for each element is denoted by $\phi_{(s)}$. It depends upon the amount α the balance wheel is turned, and also upon the conditions at the end point. In the deformed state the potential energy stored in each element is proportional to the square of the change in curvature. Since the change in angle $\phi_{(s)}$ is proportional to the square of the change in curvature. Since the change in angle $\phi_{(s)}$ is proportional to the radius of curvature for each element of the spring, the potential energy can be expressed in terms of the square of $\phi_{(s)}$. When the balance wheel is turned, the distorted shape of the spring will be such that the energy V is a minimum, subjected to the proper conditions at the two end points. From the minimization of the potential energy, it follows that $\phi_{(s)} = As + B$, where A and B are constants of integration.

For the boundary conditions of the free spring, $\phi_{(s)}$ becomes $\phi_{(s)} = -\frac{\alpha s}{L} + \alpha$. This expresses that the change in curvature for the free spring is the same everywhere along the spring, and is proportional to the angle of rotation of the balance wheel. This result substituted into the potential energy equation gives $V = \frac{\xi}{2} \frac{\alpha^2}{L}$. This indicates that

simple harmonic motion results if the torque is released; because, any motion for which the potential energy is proportional to the displacement squared, is simple harmonic motion.

The important formula arising from the consideration of the end point displacement is

$$\Delta = i\frac{\alpha}{\bar{L}} e^{i\alpha} \int_0^{\bar{L}} z e^{-i\frac{\alpha s}{\bar{L}}} ds .$$

This result is valid both for the free end and the fixed end spring. The problem consists in designing the undistorted spring so that the end point displacement, and hence, the integral is as small as possible.

For the spring with a fixed end point the square of the reaction force is

$$\lambda^2 + \mu^2 \simeq \frac{|\Delta|^2}{A^2 \bar{L}^2} .$$

This proves that the reaction force is approximately proportional to the displacement.

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The important formula arising from the consideration of the end point displacement is

$$\Delta = \frac{c_4}{L} \int_0^L z\, G - c_4 s \, ds .$$

This result is valid both for the free end and the fixed end spring. The problem consists in designing the undistorted spring so that the end point displacement, and hence, the integral is as small as possible.

For the spring with a fixed end point the square of the reaction force is

$$f_s^2 + M_s^2 = \frac{|\Delta|^2}{A^2 L^2} .$$

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